Thin-film free-surface flows

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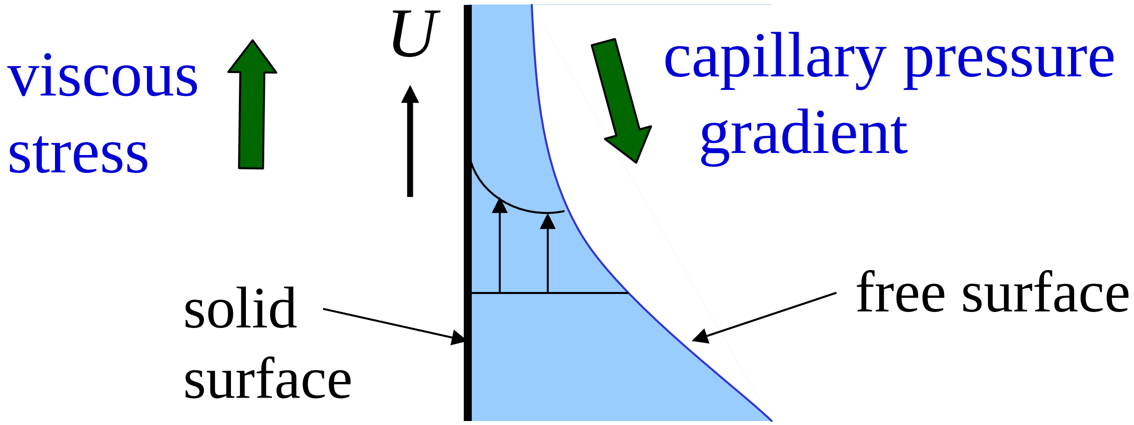
Fundamentals of coating flows:
interaction of viscous stresses & capillary stresses (due to curvature gradients)



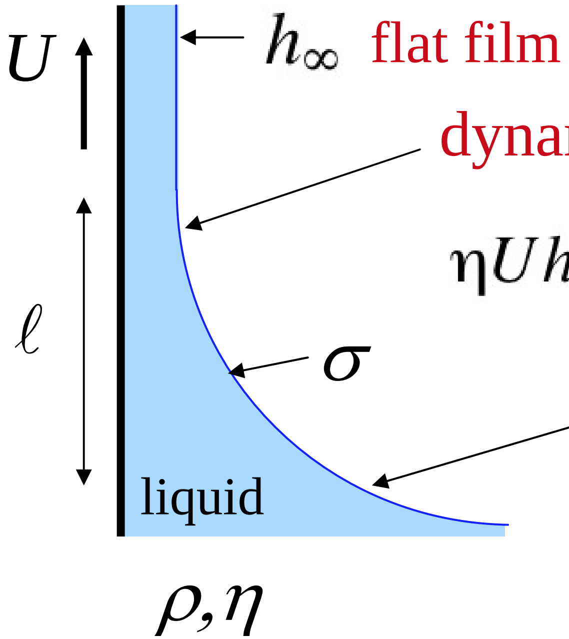
low Reynolds # Newtonian coating flows characterized by capillary number:

$$C = \frac{\eta U}{\sigma} \qquad \frac{\text{viscous stress}}{\text{capillary stress}}$$

$\eta =$ viscosity $\quad U =$ characteristic velocity $\quad \sigma =$ surface tension

The Landau-Levich-Derjaguin scaling

Landau & Levich (1942), Derjaguin (1943)



dynamic meniscus: viscous force balances surface tension

$$\eta U h_\infty \approx \sigma h_\infty^3 \frac{d\kappa}{dx} \approx \sigma \frac{h_\infty^4}{\ell^3}$$

static meniscus: gravity balances surface tension

$$\frac{h_\infty}{\ell^2} \approx \kappa_m \qquad \text{(meniscus curvature)}$$

asymptotic matching of the surface curvature in the two regions:

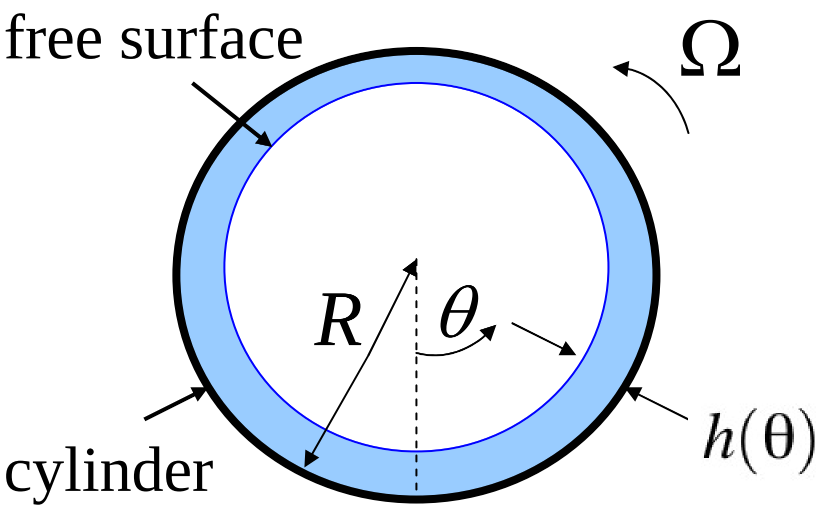
$$h_\infty = 1.34 \left(\frac{\eta U}{\sigma} \right)^{2/3} \kappa_m^{-1}$$

provided $\frac{\eta U}{\sigma} \ll 1$

(low capillary number limit)

Rimming flows:
coating the inside of a rotating cylinder

with Anette Hosoi *(MIT)* & Howard A Stone *(Harvard University)*



$A =$ filling fraction $(0 \leq A \leq 1)$

$\rho =$ density

$\eta =$ viscosity

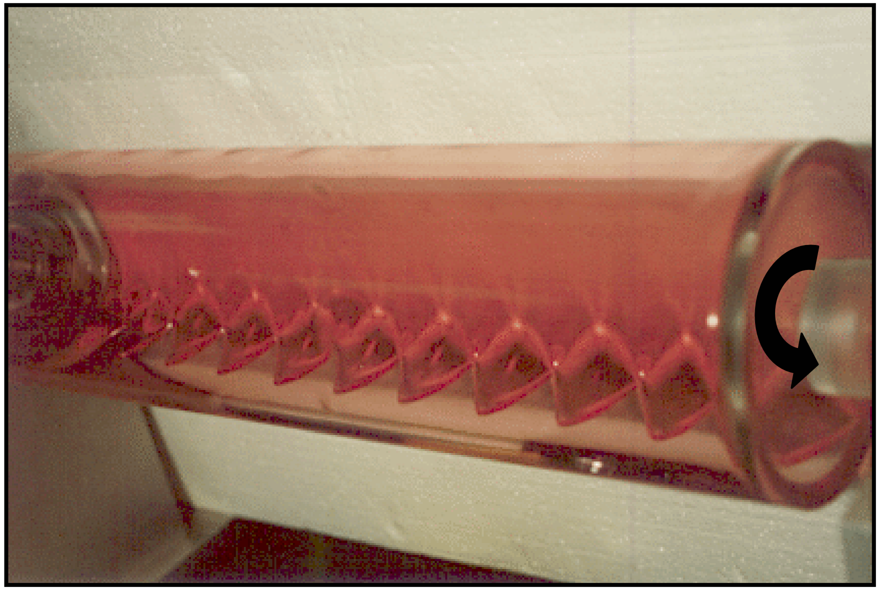
$\sigma =$ surface tension

equation for $h(\theta)$ is usual coating flow equation, but this problem has two unusual features:

1) solutions must be periodic

2) conservation of mass imposes integral BC

A rich variety of behavior …

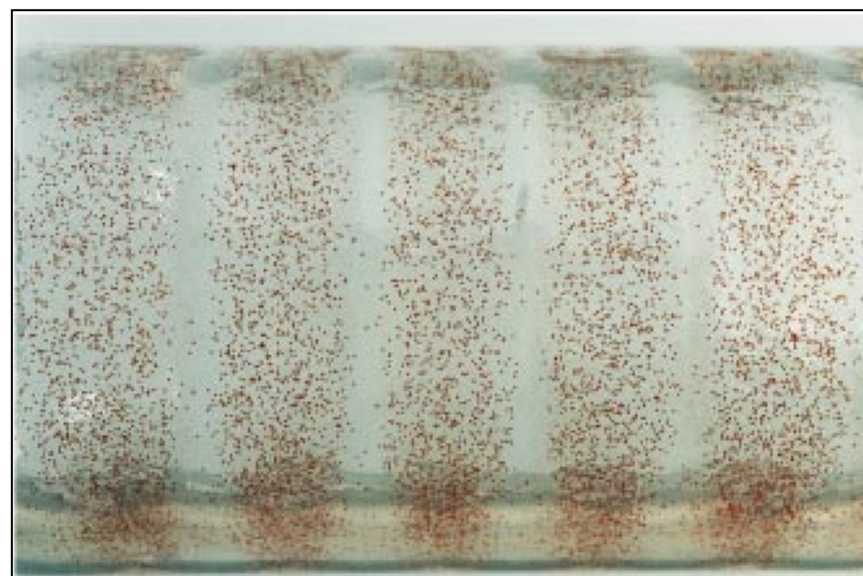


“sharks’ teeth” with pure fluid

(Thoroddsen & Mahadevan, 1997)

banding of a suspension

(Tirumkudulu, Mileo & Acrivos, 2000)



Literature review
(not comprehensive)

- films *outside* a rotating cylinder: Moffatt (1977)
- analyses in the *absence of surface tension*:

  e.g. O'Brien & Gath (1988), Johnson (1988), Wilson & Williams (1997)
- surface tension included in numerical studies:

  e.g. Hosoi & Mahadevan (1999), Tirumkudulu & Acrivos (2001)
- an unpublished study including analytical work:

  Benjamin et al. (analytical, numerical and experimental)

no previous detailed analytical study of surface tension effects

Focus of our study: surface tension effects

- two-dimensional (axially uniform) steady states
- consider significance of surface tension in “slow rotation” limit
- compare theoretical predictions & numerical results

surface tension is a singular perturbation

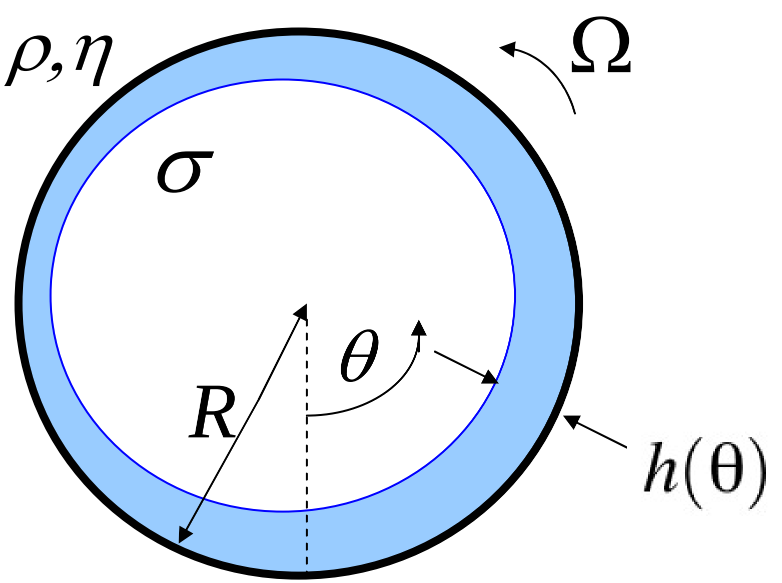
Assumptions

- lubrication approximation

$A \ll 1$

- negligible inertia

$$\frac{A^2 \rho \Omega R^2}{\eta} \ll 1$$



filling fraction A

Nondimensional flux equation:

determines film thickness $h(\theta)$

$$q = h - \frac{\lambda}{3} h^3 \sin\theta + \frac{\lambda}{3B} h^3 \frac{d\kappa}{d\theta} - \frac{A\lambda}{3} h^3 \frac{dh}{d\theta} \cos\theta + O(A, A\lambda B^{-1}, A^2\lambda)$$

- q: constant flux, to be determined
- h: viscous
- $\frac{\lambda}{3} h^3 \sin\theta$: gravity
- $\frac{\lambda}{3B} h^3 \frac{d\kappa}{d\theta}$: surface tension
- $\frac{A\lambda}{3} h^3 \frac{dh}{d\theta} \cos\theta$: "higher order" gravity

nonlinear third-order differential equation

boundary condition $\int_0^{2\pi} h(\theta)\, d\theta \approx \pi$

$\kappa =$ curvature

3 nondimensional parameters when inertia is neglected:

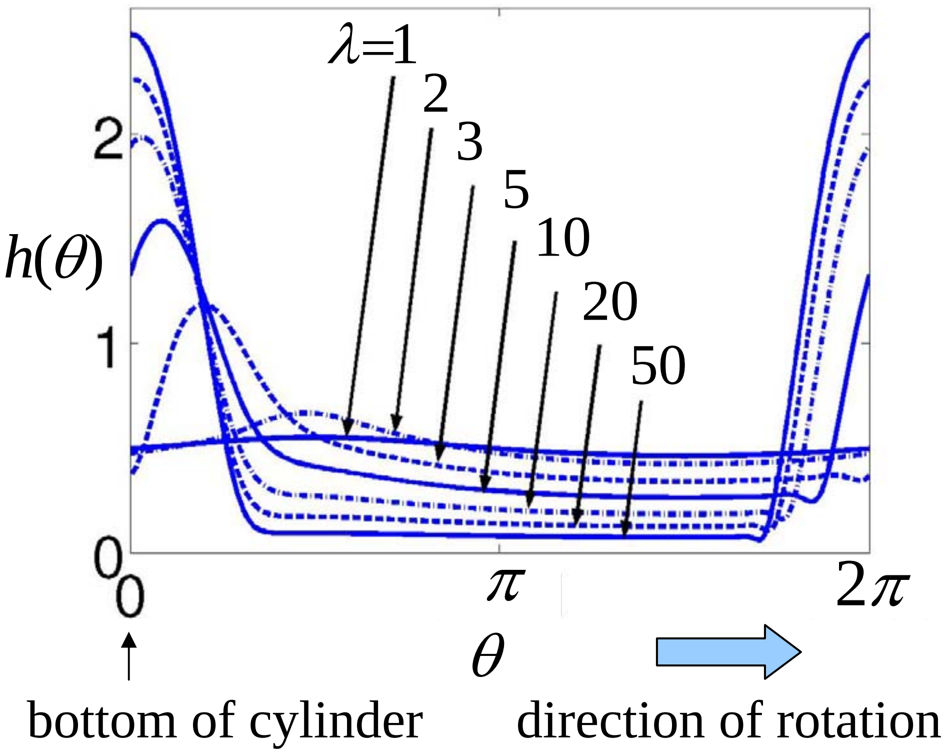
$$\lambda = \frac{A^2 \rho g R}{\eta \Omega} = \frac{\text{gravity}}{\text{viscous}}$$

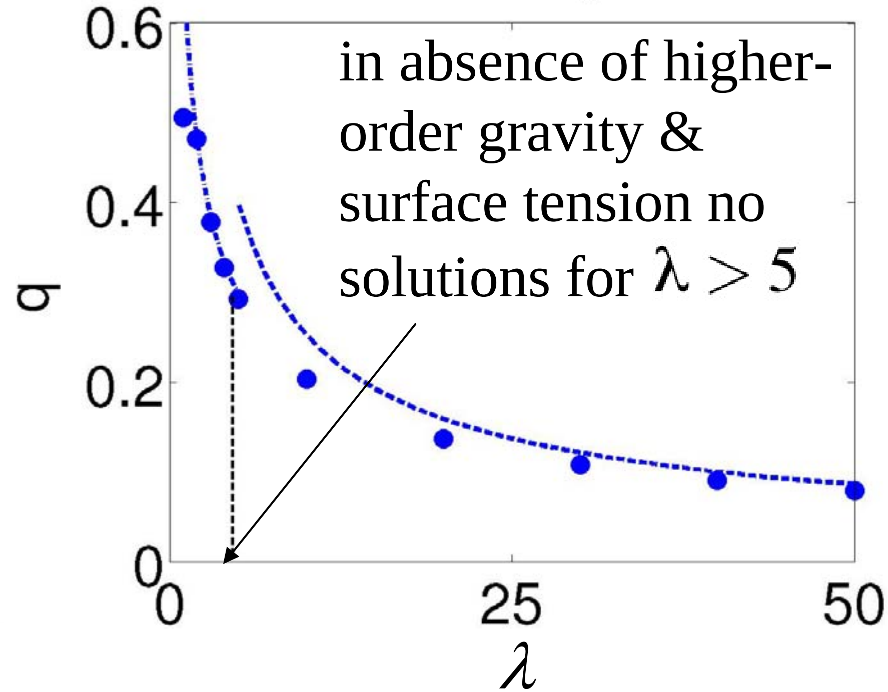
$$B = \frac{\rho g R^2}{A\sigma} = \frac{\text{gravity}}{\text{surface tension}}$$

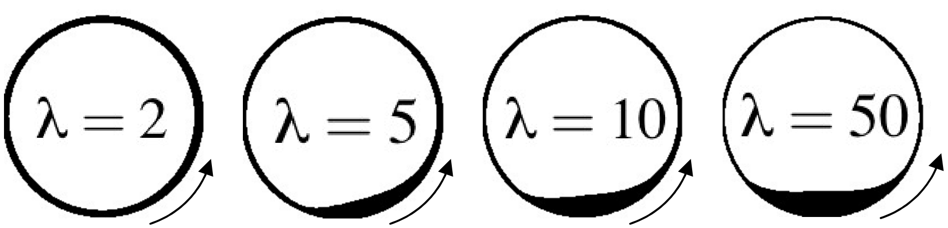
$A =$ filling fraction

(note $\lambda B^{-1} \propto C^{-1}$)

Dependence of solutions on $\lambda = \frac{A^2 \rho g R}{\eta \Omega}$



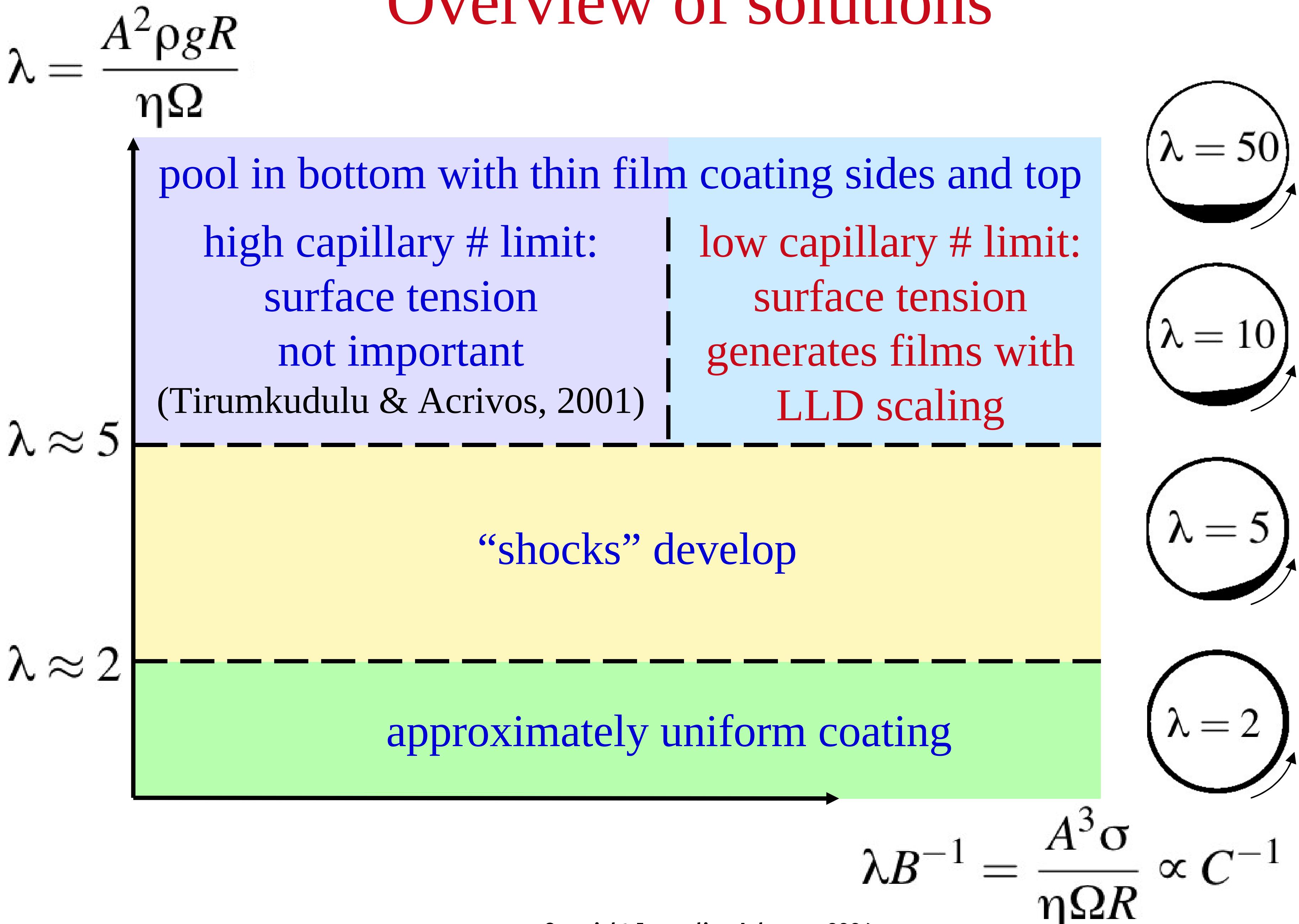




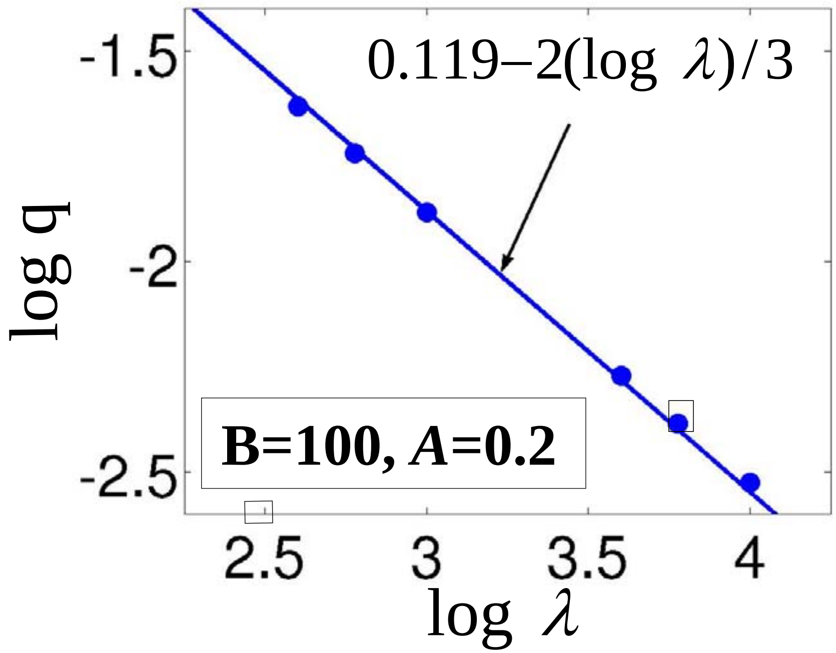
points denote value of flux from numerical simulations in low capillary # limit; B=100, A=0.1

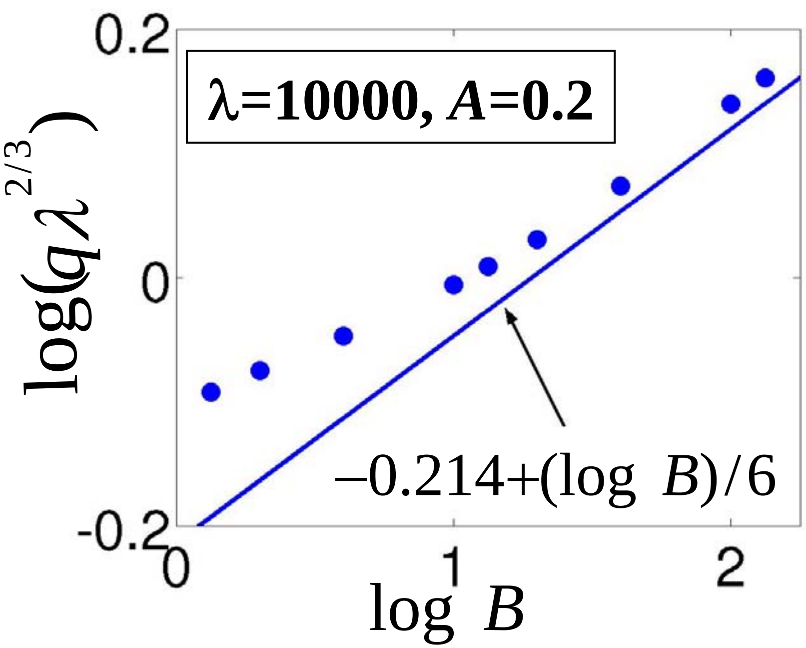
3 qualitatively different regimes: $0 < \lambda < 2$ (uniform coating); $2 < \lambda < 5$ ("shocks"); $\lambda > 5$ (pool & thin film)

Overview of solutions



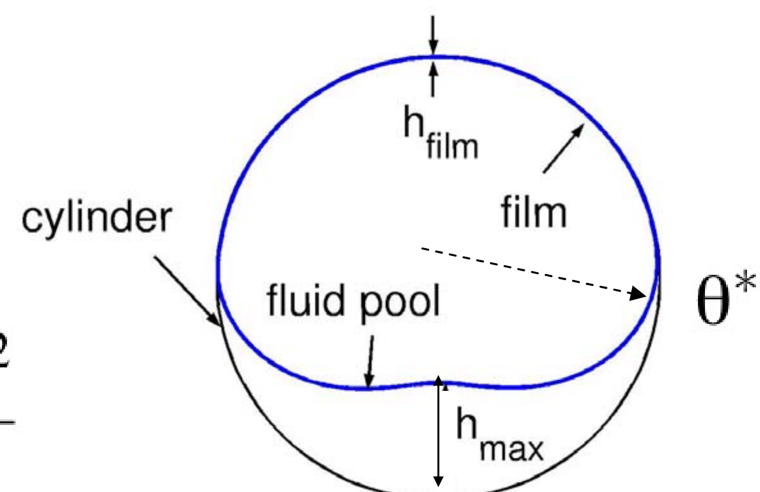
Film thickness when $\lambda = \frac{A^2 \rho g R}{\eta \Omega} > 5, B^{-3/5} \ll A \ll 1$





theoretical prediction:

$$h_{film} = \frac{0.946}{(1-\cos\theta^*)^{1/2}} \frac{A^{1/2} B^{1/6}}{\lambda^{2/3}}$$



scaling of pool determined from static considerations

$$B = \frac{\rho g R^2}{A \sigma}$$

Coating the inside of a rotating cylinder: conclusions

- inclusion of surface tension facilitates an analytical description of a steady 2-D solution at very slow rotation rates
- at slow rotation rates, a thin film is pulled out of a pool sitting at the bottom of the cylinder
- thickness of the film in low capillary # limit calculated by asymptotic matching
- dimensional film thickness:

$$h_{film} = \frac{0.946}{(1-\cos\theta^*)^{1/2}} \frac{(\eta\Omega R)^{2/3}}{\sigma^{1/6}(\rho g)^{1/2}} = 0.798 \frac{(\eta\Omega R)^{2/3}}{A^{1/3}\sigma^{1/6}(\rho g)^{1/2}}$$

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