A New Approach to LES Modeling

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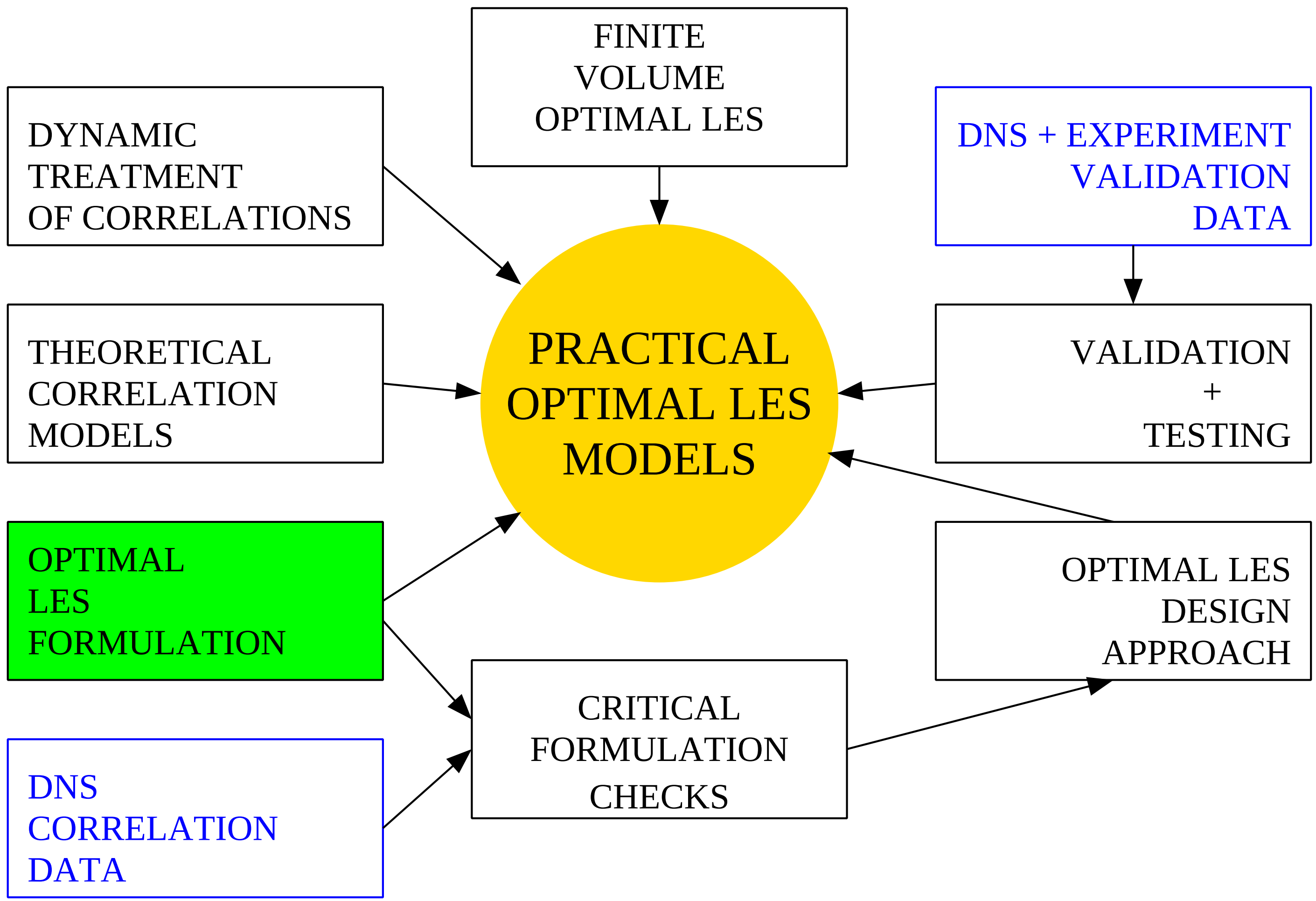
or

Optimal LES:
Trading in the Navier-Stokes Equations for Custom Designed Discrete LES

Large Eddy Simulation

- Simulate only the largest scales of High-Reynolds number turbulence
  - Models of small scales required
- Numerous models developed recently
  - E.g. scale similarity, dynamic, structure function, stretched vortex, deconvolution
- Difficulties remain
  - Wall-bounded turbulence
  - Impact of numerical discretization
- LES is for making predictions!
  - Predict (some) statistical properties of turbulence
  - Predict large-scale dynamics of turbulence

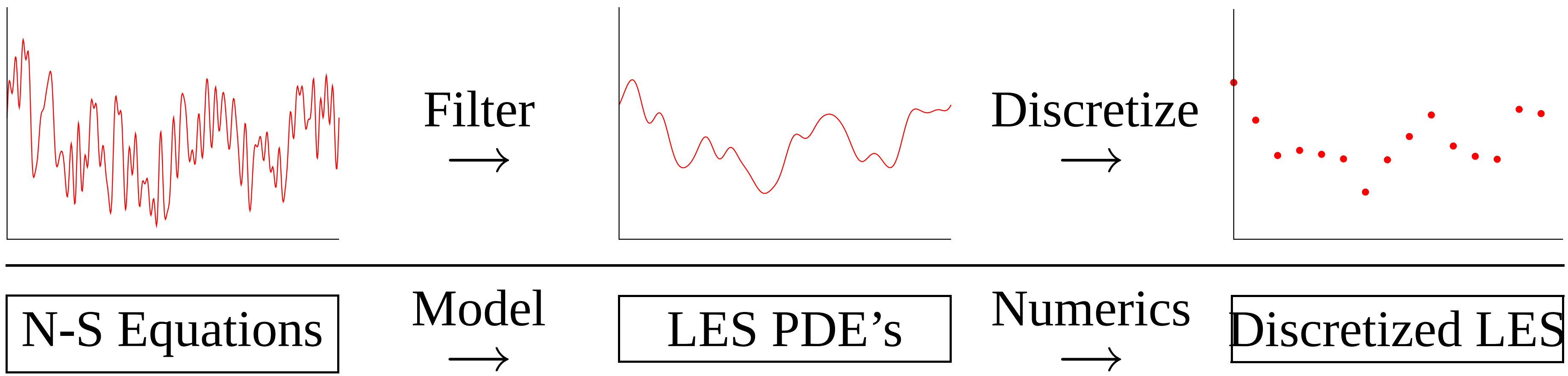
Optimal LES Development Map



Filtering and LES

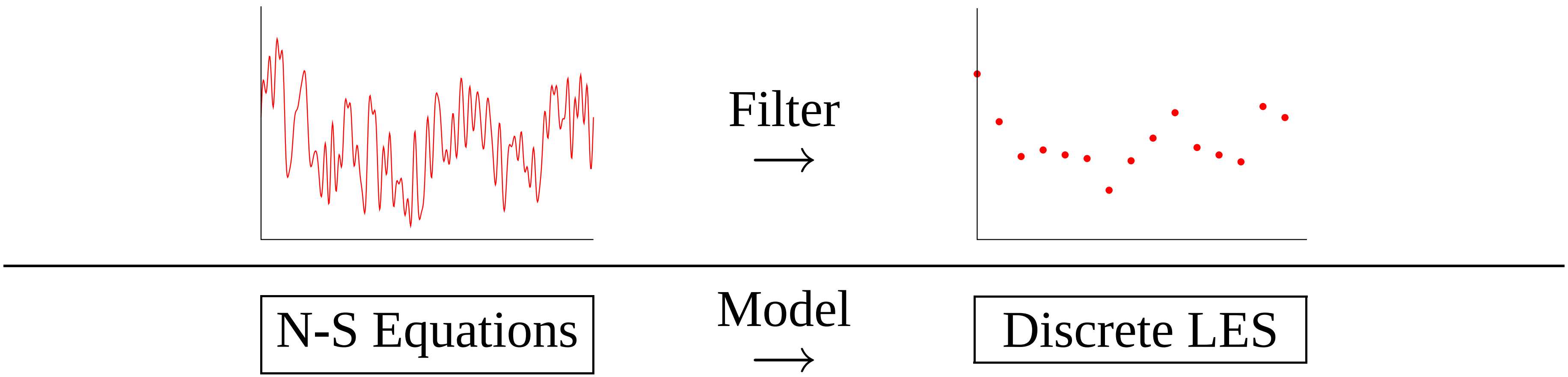
- Filters precisely define the large scales to be simulated
  - Not absolutely necessary, but useful
  - Provides a framework in which to develop models
- Two flavors of filtering and LES
  - Continuously filtered LES
  - Discretely filtered LES

Continuous LES



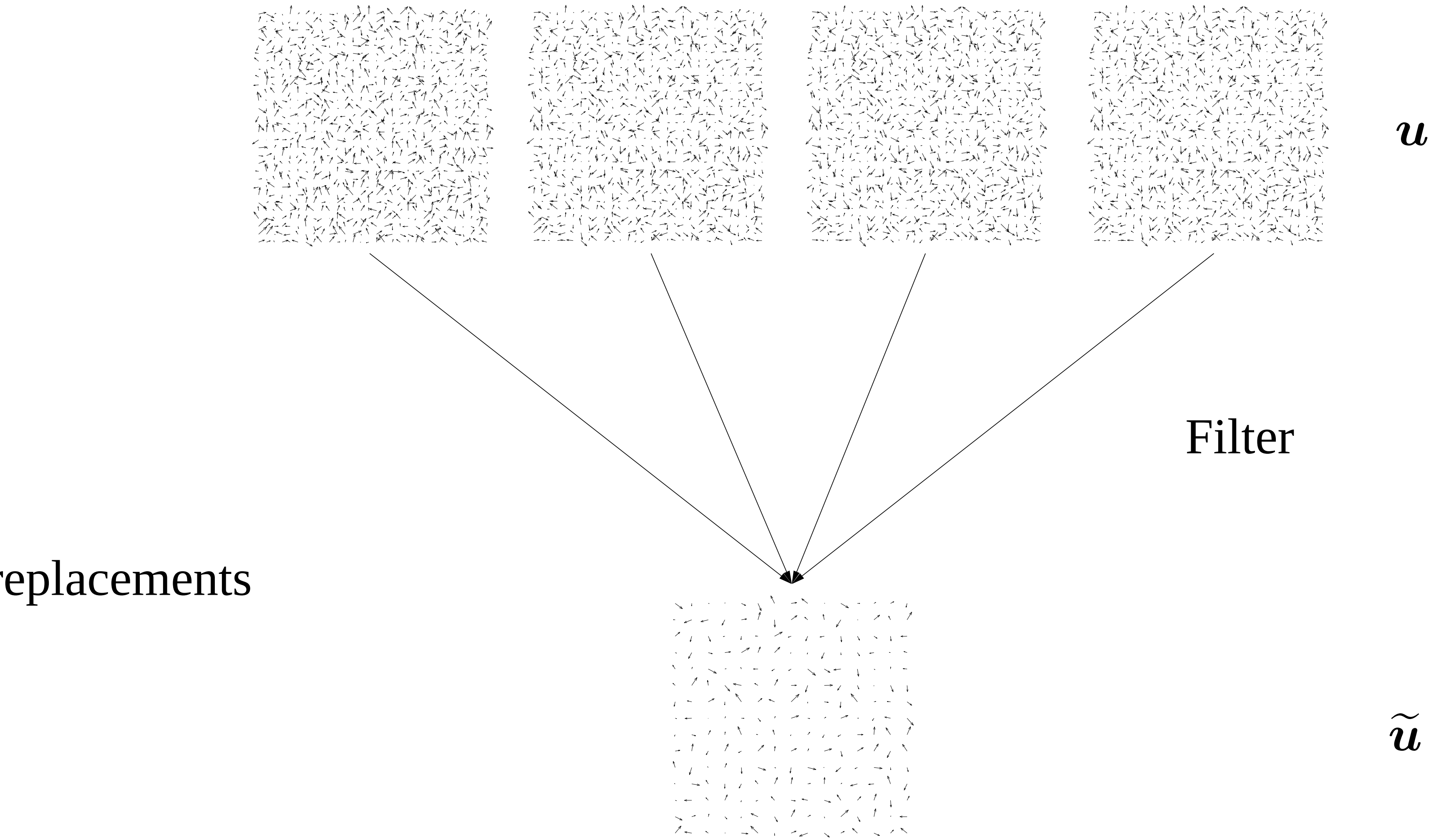
- Many filters are invertible or nearly so (e.g. Gaussian)
- A hypothetical exercise: suppose filter can be inverted
  - Determine evolution by defiltering and advancing N-S
  - Best "model" would be a DNS $\Rightarrow$ DNS resolution
  - Coarse resolution determines accuracy limits
  - Best models must depend explicitly on discretization

Discrete LES

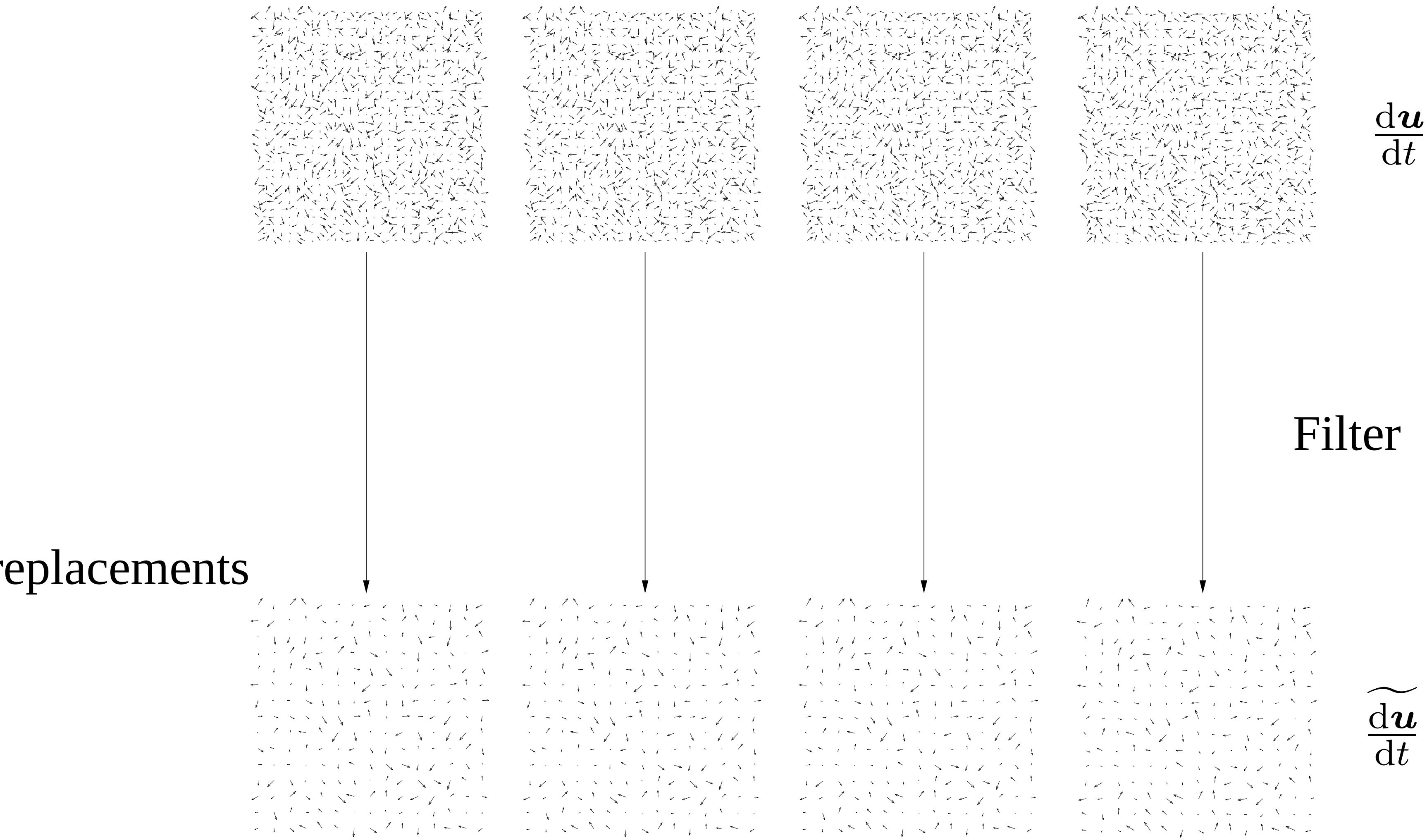


- Examples: Fourier cut-off, sampled top-hat (finite volume), MILES
- Filter is not invertible & stochastic modeling tools are applicable
  - Many turbulent fields map to same filtered state
  - Evolution of filtered state considered stochastic
- This is the formulation used here

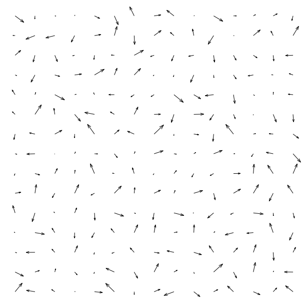
Stochastic Evolution of LES



Stochastic Evolution of LES



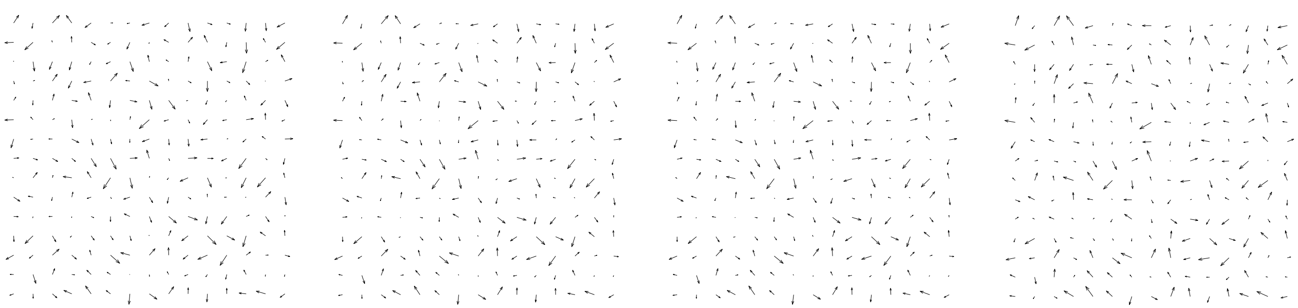
Stochastic Evolution of LES



$\widetilde{\boldsymbol{u}}$

Mapping from filtered field to filtered evolution?

replacements

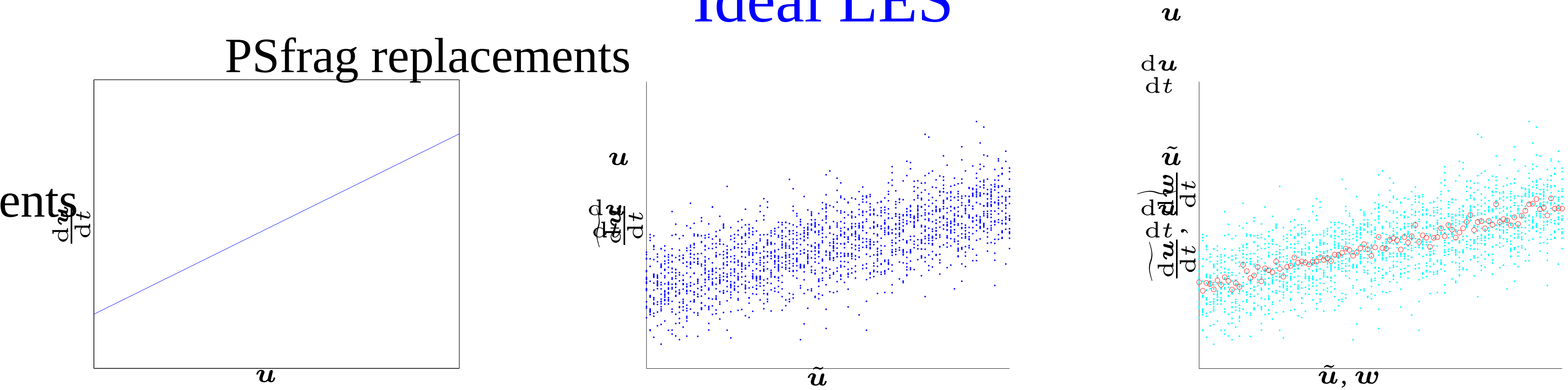


$\widetilde{\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}}$

PSfrag replacements

Ideal LES

PSfrag replacements



Turbulence　　　Filtered Turbulence　　　Ideal LES

- Best deterministic LES evolution: Average of filtered evolutions of fields mapping to the current LES state

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} = \left\langle \widetilde{\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}} \middle| \tilde{\boldsymbol{u}} = \boldsymbol{w} \right\rangle$$

  - Equivalently average of model terms: $m = \langle M | \tilde{\boldsymbol{u}} = \boldsymbol{w} \rangle$
  - Two Theorems:

    1) 1-time statistics of $\boldsymbol{w}$ and $\tilde{\boldsymbol{u}}$ match (Pope 2000, Langford & Moser 1999)

    2) Mean-square difference between $\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t}$ and $\frac{\mathrm{d}\tilde{\boldsymbol{u}}}{\mathrm{d}t}$ minimized but finite.

Optimal LES

- Statistical data requirements for Ideal LES are outrageous
  - # of conditions $=$ # DOF in LES
- Stochastic estimation as an approximation to conditional average
  - Pick functional form of $m(\boldsymbol{w})$
  - Minimize mean-square error of approximation to conditional average
  - Results in model formulation first proposed by Adrian (1979,1990)

Optimal LES

An example

- Estimate conditional average $m \approx \langle M | \tilde{u} = w \rangle$
- Suppose $m(w) = A + Bw + Cw^2 + Dw^3$, then

$$\langle (M - m(\tilde{u}))E_j \rangle = 0 \qquad \Rightarrow \qquad \langle ME_j \rangle = \langle m(\tilde{u})E_j \rangle$$

  where $E = (1, \tilde{u}, \tilde{u}^2, \tilde{u}^3)$ is the event vector

- Equations solved for coefficients A, B, C and $D \Rightarrow$ Optimal model
- Must know $\langle ME_j \rangle$ and $\langle E_i E_j \rangle$
  - Try using DNS correlation data
  - Then get correlations from theory

Ideal vs. Optimal LES

- For a given turbulent flow and filter, Ideal LES is uniquely defined but unknown
- In contrast, several choices must be made to define Optimal LES
  - Selection of modeled term M

    E.g. $\frac{\mathrm{d}\tilde{\boldsymbol{u}}}{\mathrm{d}t}$, τ_{ij} or $\partial_j \tau_{ij}$

    Matters because error minimized is different
  - Selection of model dependencies

    E.g. spatial locality, nonlinearity

    Matters because changes space in which minimum error is sought

Developing Optimal LES Models

- Modeler needs to design the Optimal model
  - Guidance provided by $\langle ME_j \rangle = \langle mE_j \rangle$
  - Arrange so $\langle ME_j \rangle$ includes terms of dynamical interest

    Model reproduces them *a priori*

    Example: Terms in 2-point correlation or Reynolds stress equation

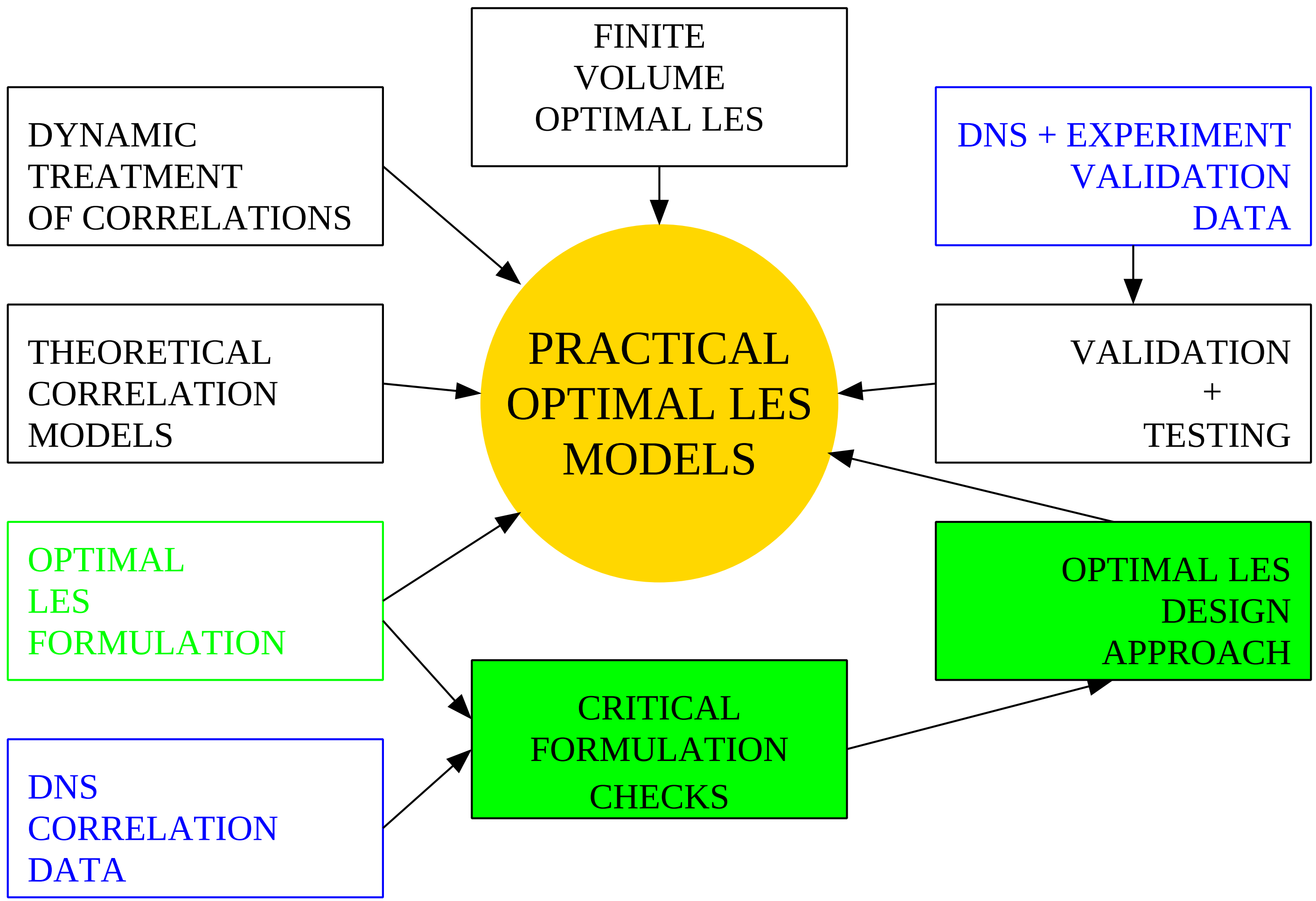
- Statistical information required as input
  - For quadratic estimates need correlations:

$$\langle u_i(\mathbf{x})u_j(\mathbf{x}')\rangle \qquad \langle u_i(\mathbf{x})u_j(\mathbf{x}')u_k(\mathbf{x}')\rangle \qquad \langle u_i(\mathbf{x})u_j(\mathbf{x}')u_k(\mathbf{x}'')u_l(\mathbf{x}'')\rangle$$

  with separations of order the non-locality of the model

- Use DNS correlations for testing,
- Theoretically determined correlations later.

Optimal LES Development Map



Tests of Optimal LES with DNS Statistical Data

- Evaluate modeling approach without other uncertainties
- Principles of Optimal model design
- Test Cases:
  - Forced isotropic turbulence ($Re_\lambda = 164$)

    Fourier cutoff filter
  - Turbulent flow in a plane channel ($Re_\tau = 590$)

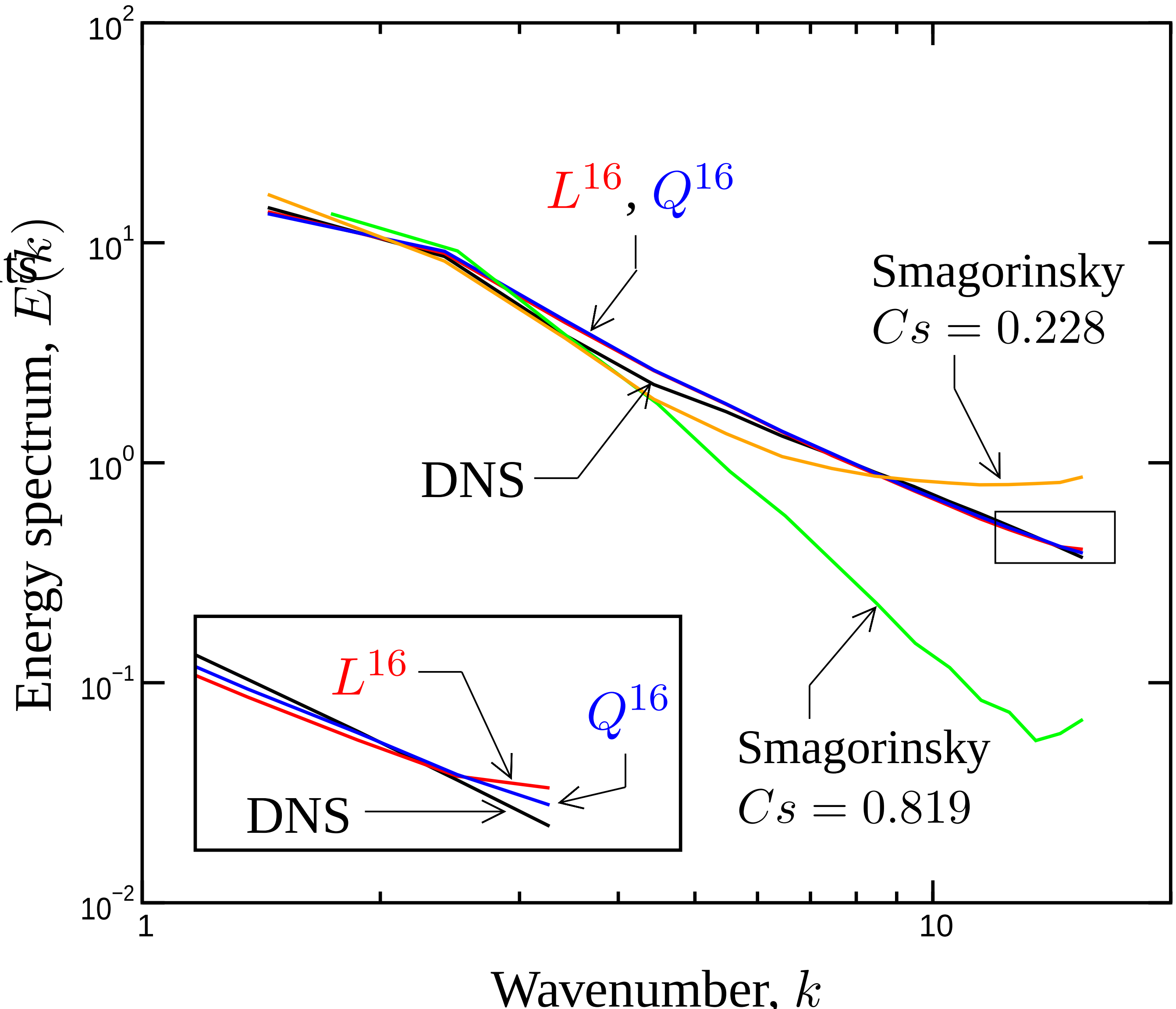
    Spectral representation/filter

    Severely filtered ($\Delta x^+ = 116$, $\Delta z^+ = 58$)
  - Forced isotropic turbulence

    Finite volume filter

Optimal LES of Forced Isotropic turbulence

g replacements



Optimal LES of Turbulent Channel at $Re_\tau = 590$

PSfrag replacements

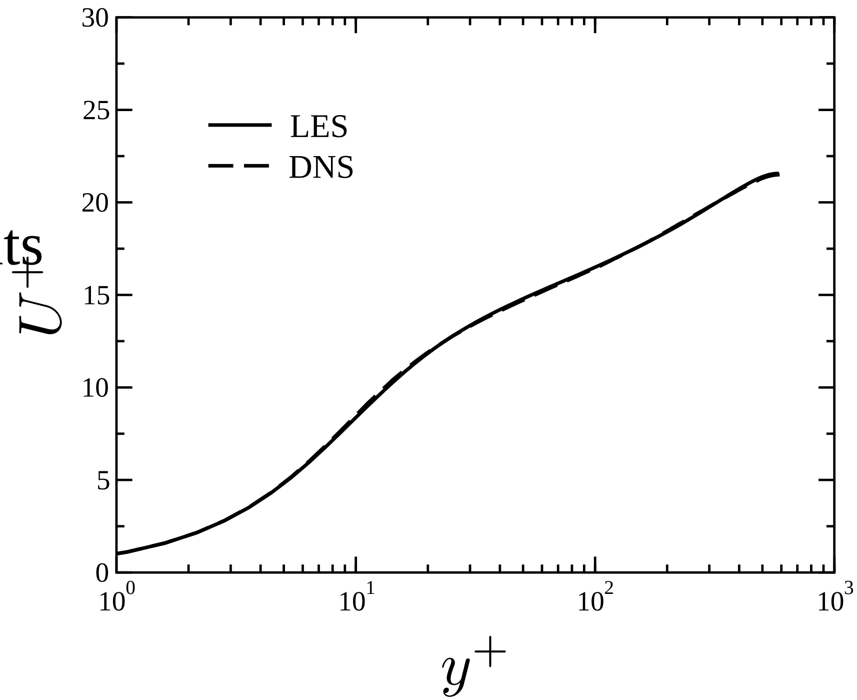
y^+

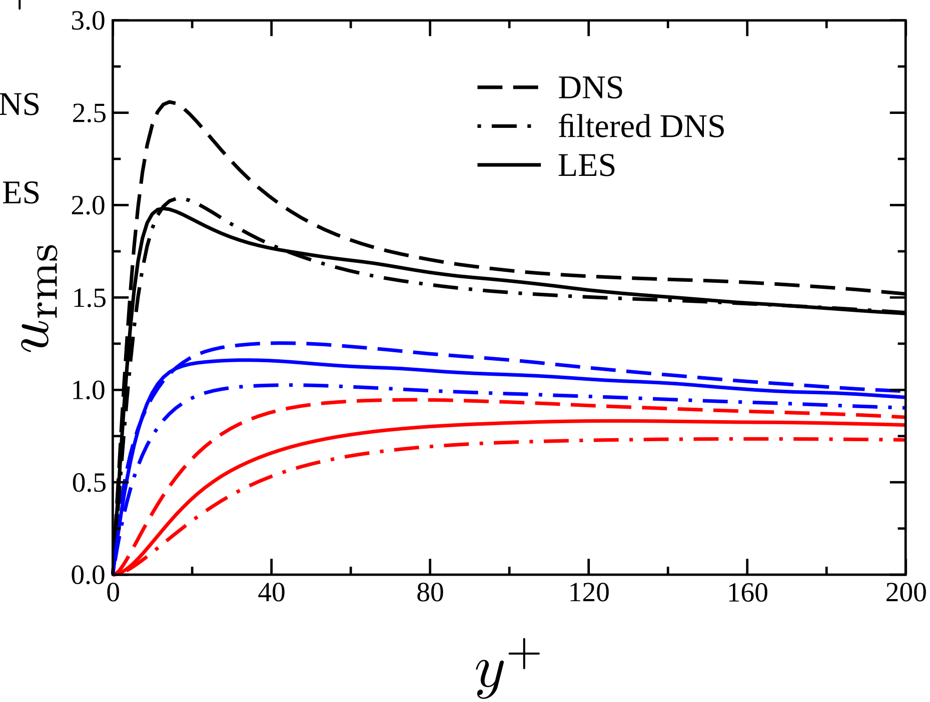
U^+

DNS

LES

ents





Constructing Good Optimal Models

- Optimal model was formulated to reproduce the y–transport term in the Reynolds stress equation $(\partial_y u_k \tau_{i2})$

PSfrag replacements

- Simpler optimal model that doesn't reproduce transport yields:

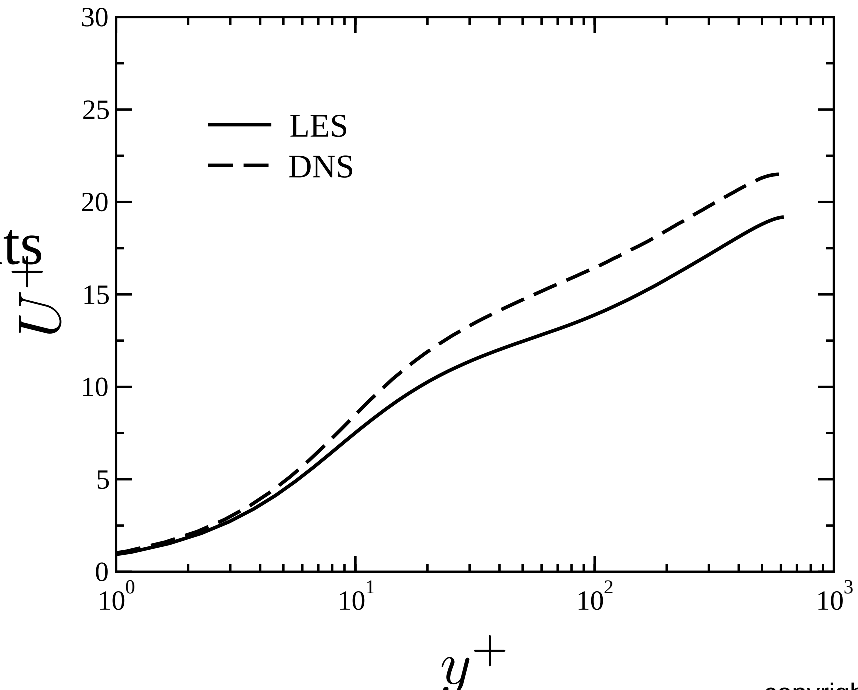
y^+

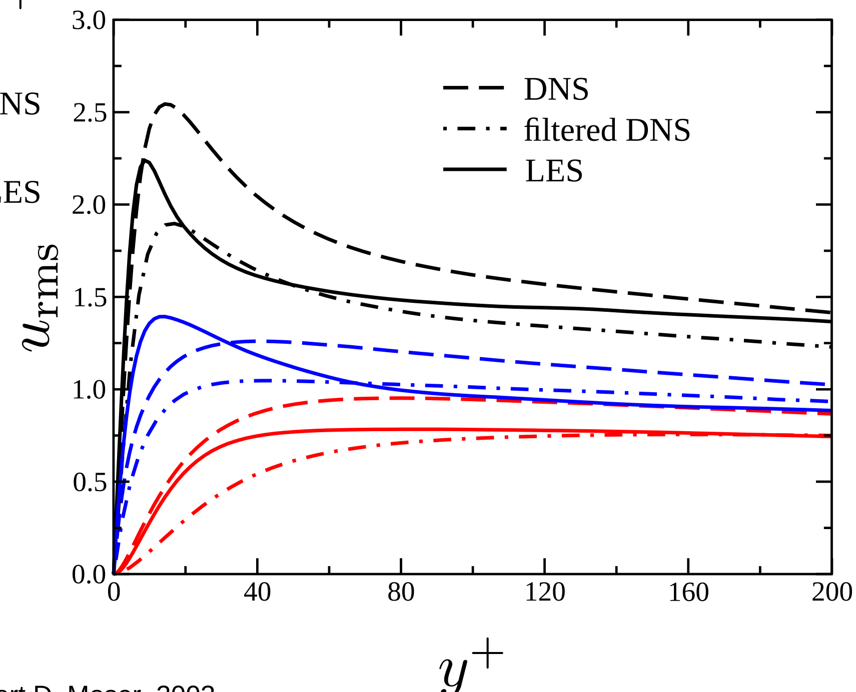
U^+

DNS

LES

ments

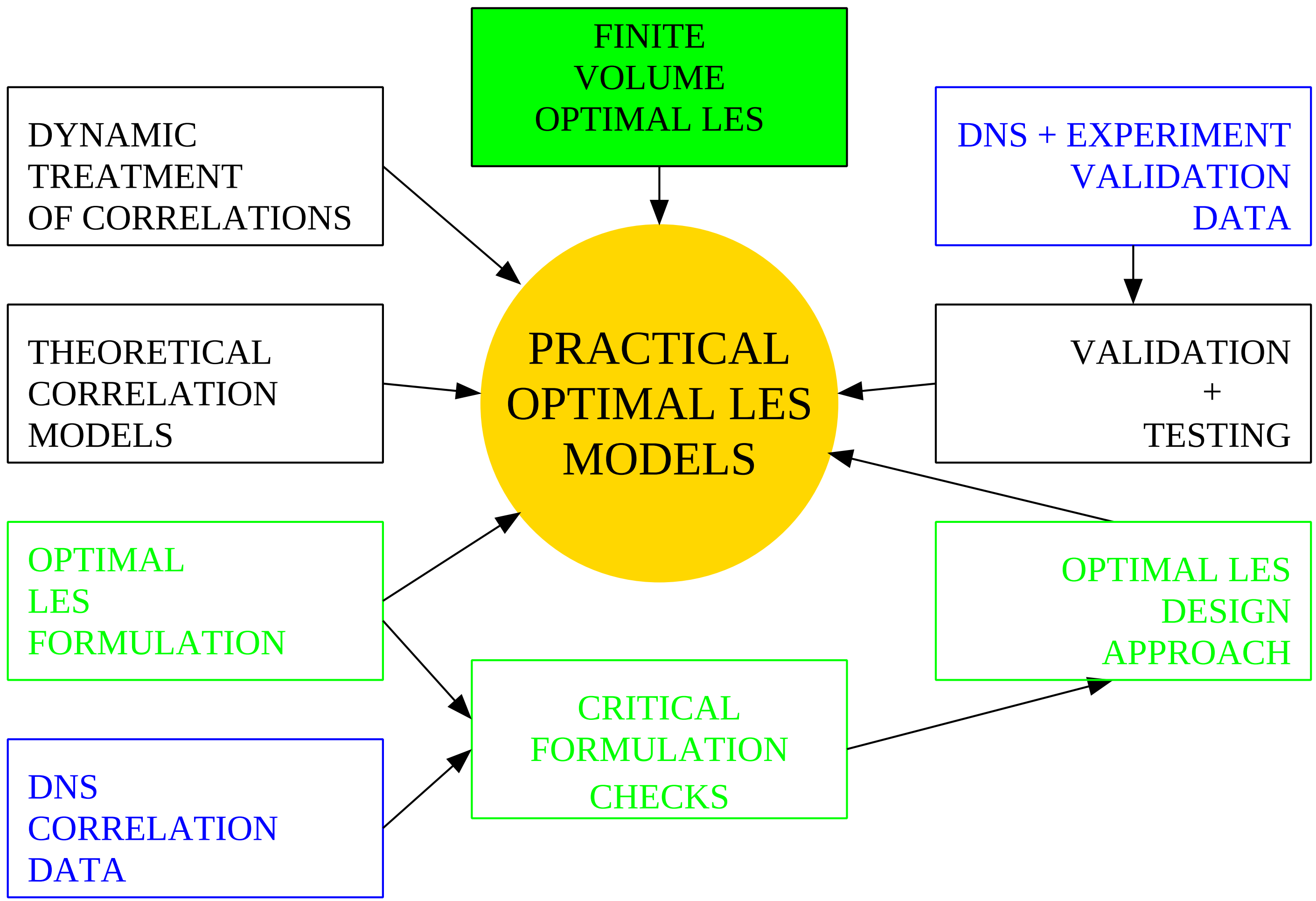




Responsibilities of an LES Model

- An LES model must represent several effects of the subgrid turbulence
  - Dissipation of energy (and R_{ij}) - standard requirement
  - Subgrid contribution to mean equation (unresolved Reynolds stress).
  - Subgrid contribution to R_{ij} transport
  - Subgrid contribution to pressure redistribution of R_{ij}
- Optimal LES provides a mechanism to construct models that do this
  - Select M and E_j, so that $\langle M E_j \rangle$ includes terms in R_{ij} equation

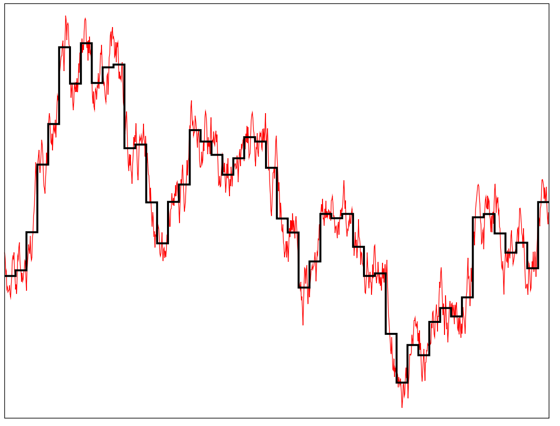
Optimal LES Development Map



Finite Volume Optimal LES

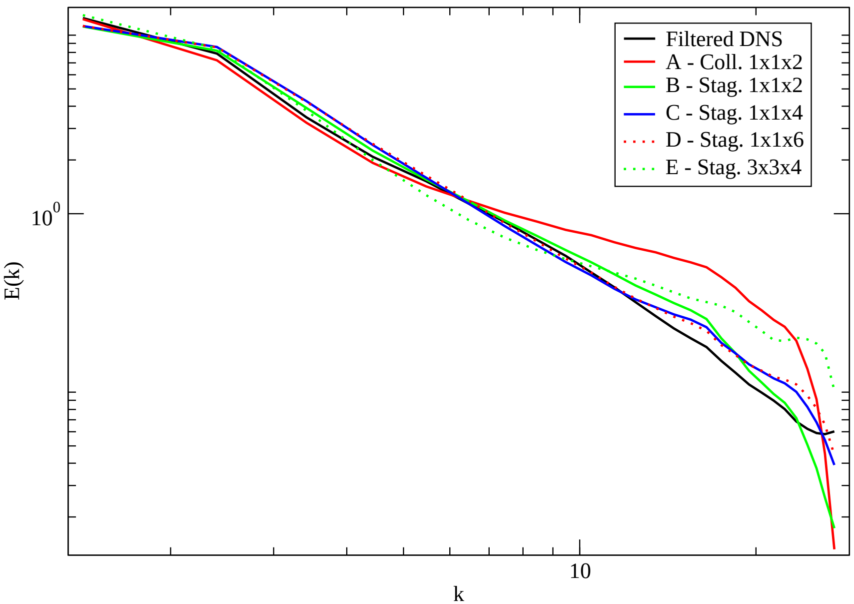
- Like standard finite volume schemes, except:
  - Cell size not small compared to turbulence scales
  - Standard reconstruction techniques to determine finite volume fluxes are not applicable

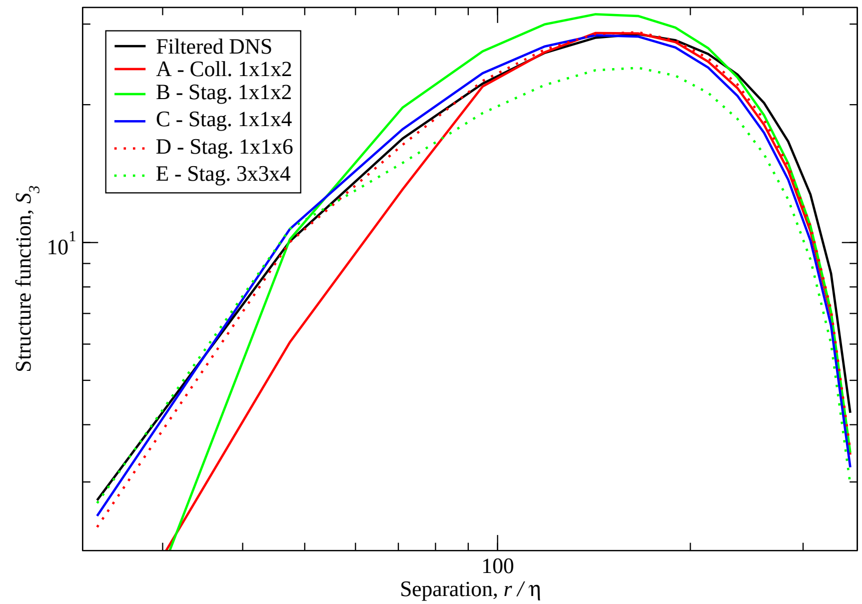
    True solution is not smooth on scale of the grid volume.
- Fluxes must be modeled.
  - Use Optimal model of the fluxes.
  - Estimate consistent with turbulence statistics, not numerical convergence.
- Need FV formulation for complex geometries
- Similar approach for Finite Difference and Finite Element discretizations



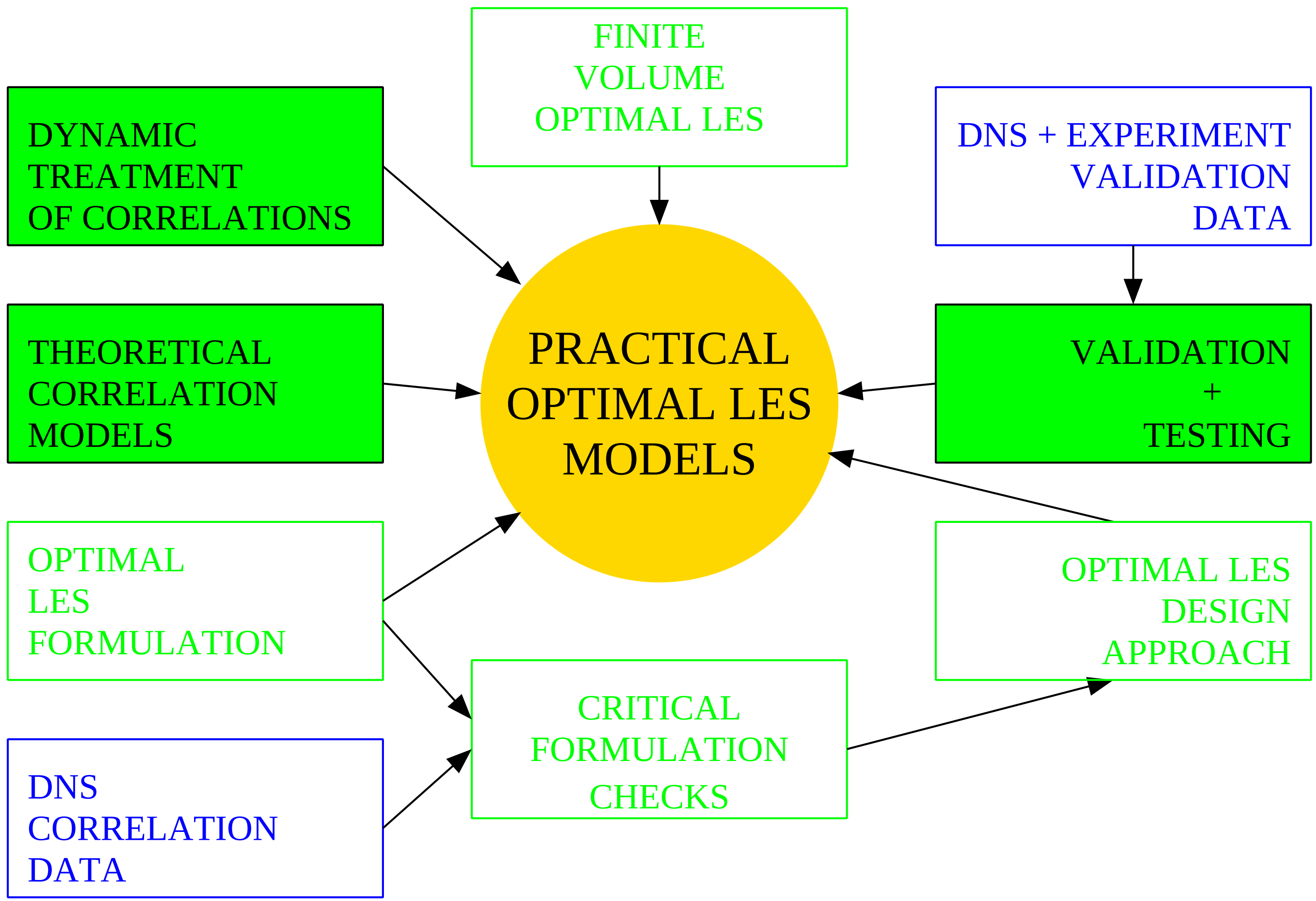
Performance of FV LES, $Re_\lambda = 164$

32^3 Isotropic LES





Optimal LES Development Map



Making Optimal LES Useful

- Simulations shown so far relied on DNS statistical data
  - Allowed properties and accuracy of OLES models to be explored
  - Allowed formulation details to be determined
  - Has not produced useful models
    - Need to do a DNS first
- Statistical input is needed
  - Rely as much as possible on theory

High Reynolds Number Optimal FV LES

- Estimation equations are of the form:

$$
\begin{aligned}
M'_{ij} &= \sum_{\alpha} L^{\alpha}_{ijk} w^{\alpha}_{k} + \sum_{\alpha,\beta} Q^{\alpha\beta}_{ijkl} (w^{\alpha}_{k} w^{\beta}_{l})' \\
\langle w^{\gamma}_{m} M'_{ij} \rangle &= \sum_{\alpha} L^{\alpha}_{ijk} \langle w^{\alpha}_{k} w^{\gamma}_{m} \rangle + \sum_{\alpha,\beta} Q^{\alpha\beta}_{ijkl} \langle (w^{\alpha}_{k} w^{\beta}_{l})' w^{\gamma}_{m} \rangle \\
\langle (w^{\gamma}_{m} w^{\delta}_{n})' M'_{ij} \rangle &= \sum_{\alpha} L^{\alpha}_{ijk} \langle w^{\alpha}_{k} (w^{\gamma}_{m} w^{\delta}_{n})' \rangle \\
&\quad + \sum_{\alpha,\beta} Q^{\alpha\beta}_{ijkl} \langle (w^{\alpha}_{k} w^{\beta}_{l})' (w^{\gamma}_{m} w^{\delta}_{n})' \rangle
\end{aligned}
$$

- Green terms are correlations of LES variables

  Can compute from LES “on the fly” (dynamically)

- Red terms require modeling input

Modeling the Red Terms

- The red correlations are surface/volume integrals of:

$$\langle u_i(\mathbf{x})u_j(\mathbf{x})u_m(\mathbf{x}')\rangle \qquad \langle u_i(\mathbf{x})u_j(\mathbf{x})u_m(\mathbf{x}')u_n(\mathbf{x}'')\rangle$$

- Assume $Re \to \infty$, separations in inertial range ($\mathbf{r} = \mathbf{x} - \mathbf{x}'$)
- Small-scale isotropy, Kolmogorov $\frac{2}{3}$ and $\frac{4}{5}$ laws, Quasi-normal approximation

$$\begin{aligned}
\langle u_i(\mathbf{x})u_j(\mathbf{x}')\rangle &= u^2\delta_{ij} + \frac{C_1}{6}\epsilon^{2/3}r^{-4/3}(r_ir_j - 4r^2\delta_{ij}) \\
\langle u_i(\mathbf{x})u_j(\mathbf{x})u_m(\mathbf{x}')\rangle &= \frac{\epsilon}{15}\left(\delta_{ij}r_m - \tfrac{3}{2}(\delta_{jm}r_i + \delta_{im}r_j)\right) \\
\langle u_i(\mathbf{x})u_j(\mathbf{x})u_m(\mathbf{x}')u_n(\mathbf{x}'')\rangle &= \langle u_i(\mathbf{x})u_j(\mathbf{x})\rangle\langle u_k(\mathbf{x}')u_l(\mathbf{x}'')\rangle \\
&+ \langle u_i(\mathbf{x})u_k(\mathbf{x}')\rangle\langle u_j(\mathbf{x})u_l(\mathbf{x}'')\rangle \\
&+ \langle u_i(\mathbf{x})u_l(\mathbf{x}'')\rangle\langle u_k(\mathbf{x}')u_l(\mathbf{x})\rangle
\end{aligned}$$

Theoretical Optimal LES

- Forced Isotropic Turbulence
  - DNS at $Re_\lambda = 164$
  - LES at $Re_\lambda = \infty$

PSfrag replacements

filtered DNS, $Re_\lambda = 164$

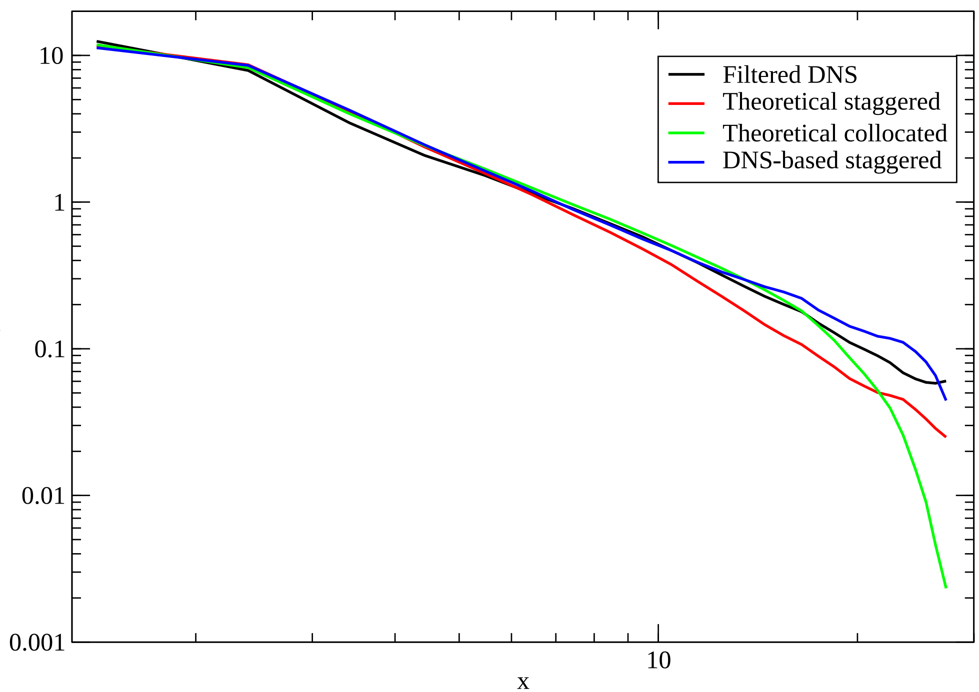
NS-based optimal LES, $Re_\lambda = 164$

Theoretical optimal LES, $Re_\lambda = \infty$

heoretical optimal LES, $Re_\lambda = 164$

k

$E(k)$



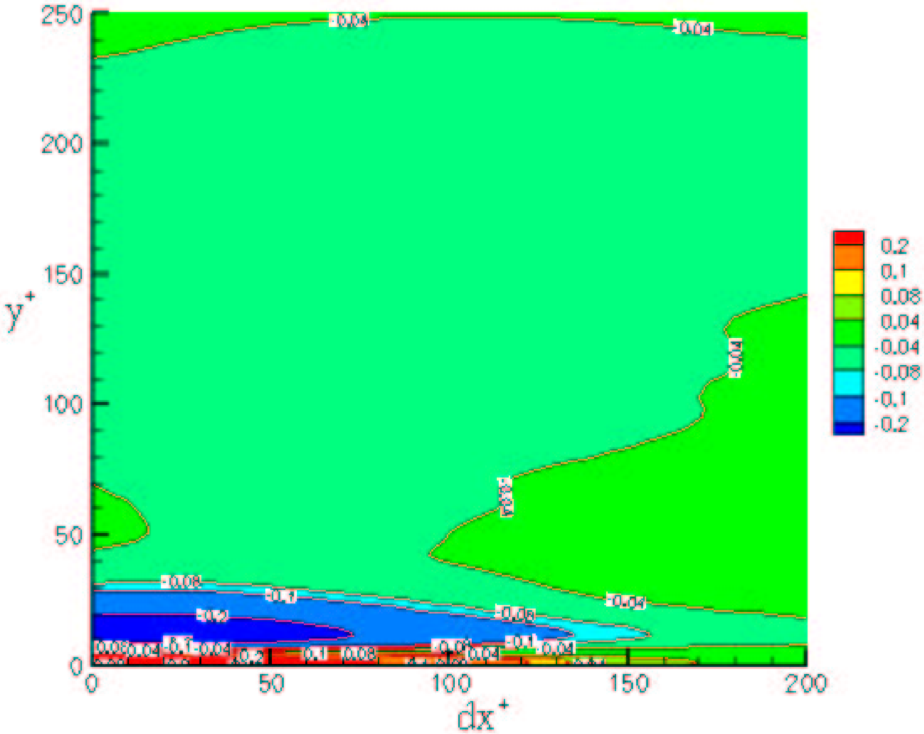
High (∞) Re Wall-Bounded Turbulence

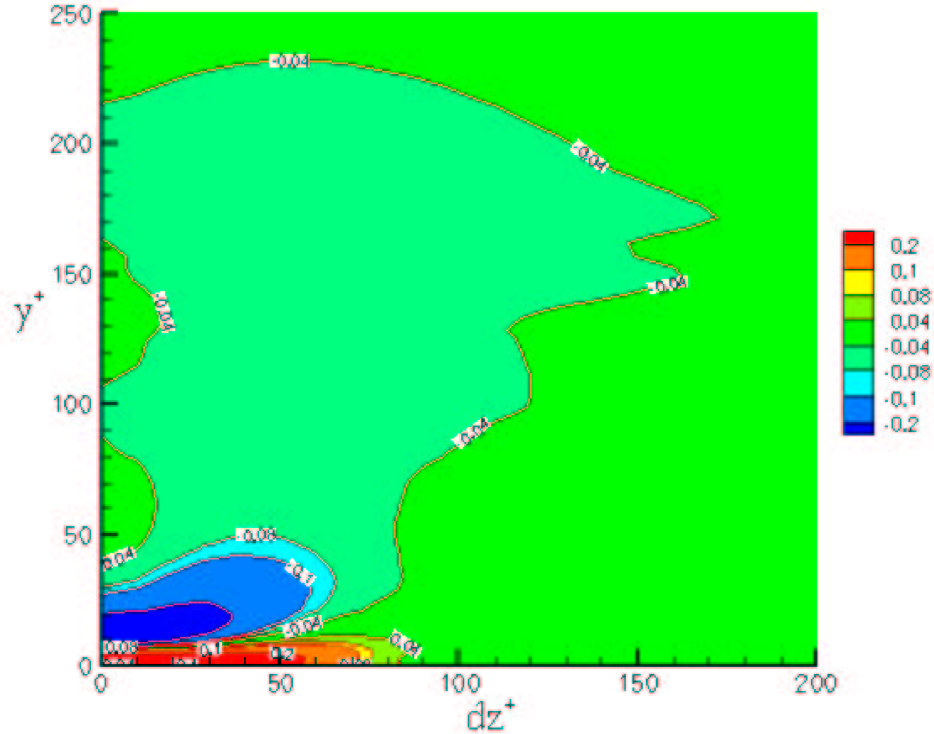
- Assumptions of isotropy and inertial range not valid near wall
- Green terms can still be determined dynamically
- Need red correlations:
- A variety of modeling tools are being evaluated:
  - Log-layer similarity (Oberlack)
  - Anisotropy expansion & scaling (Procaccia)
  - Constraints from N-S equations
  - Quasi-Normal approximation

Test of Quasi-Normal Approximation

Channel Flow at $Re_\tau = 590$

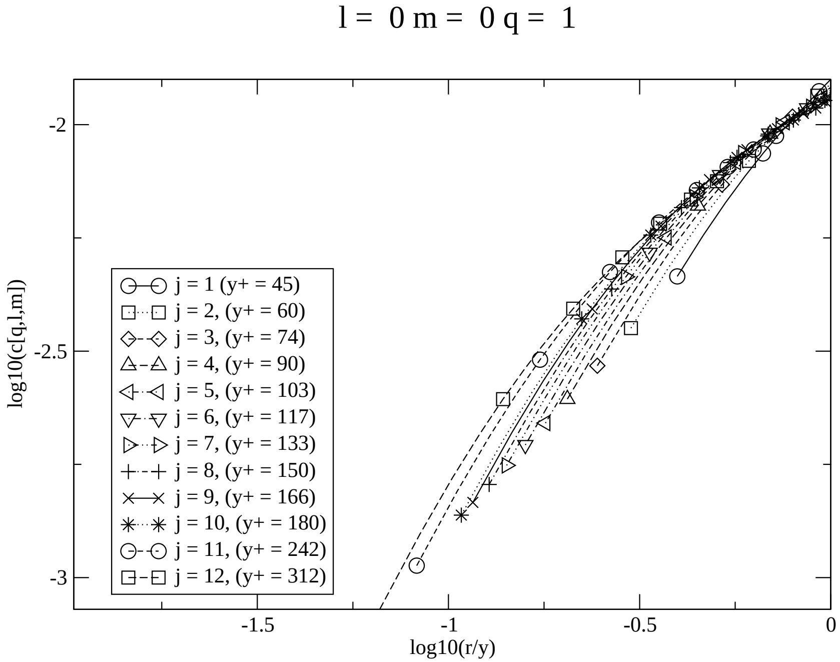
- Normalized error, $\phi_{11,11}(\mathbf{r}) = \frac{Q_{11,11}(\mathbf{r}) - QNA}{L(\mathbf{r})}$ where $L(\mathbf{r}) = \langle Q_{pq,rs}(\mathbf{r}) Q_{pq,rs}(\mathbf{r}) \rangle^{1/2}$

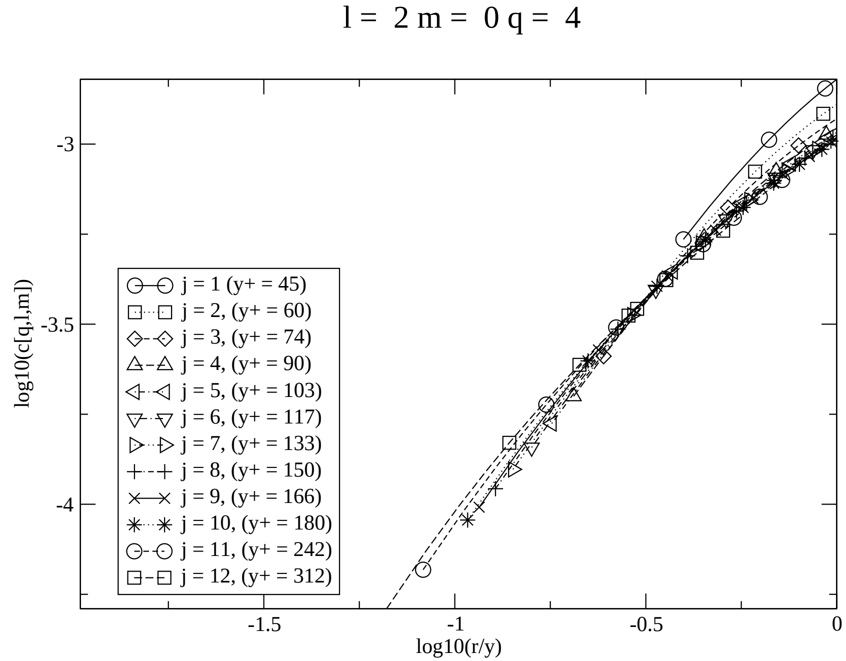




Similarity Scaling of Expansion Coefficients

in Channel at $Re_\tau = 940$, Expansion of Procaccia





Conclusions

- Discrete LES formulations are useful, avoid problems with discretization
- Optimal LES is a rational basis for discrete LES modeling
  - Yields remarkably good LES
  - But needs extensive statistical data as input
- For $Re \to \infty$, correlations available theoretically (away from walls)
  - Kolmogorov theory, Quasi-normal approximation, small-scale isotropy & a dynamic procedure.
- Near walls, need more information
- Also need models for subgrid contribution to statistical quantities of interest (e.g. turbulent energy).