Symmetries, Clusters, and Synchronization Patterns in Complex Networks

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Sponsors

- Office of Naval Research:
  - UM ... cs for No
  - DU

“O.K., let’s slowly lower in the grant money.”

Contributors and Co-Authors

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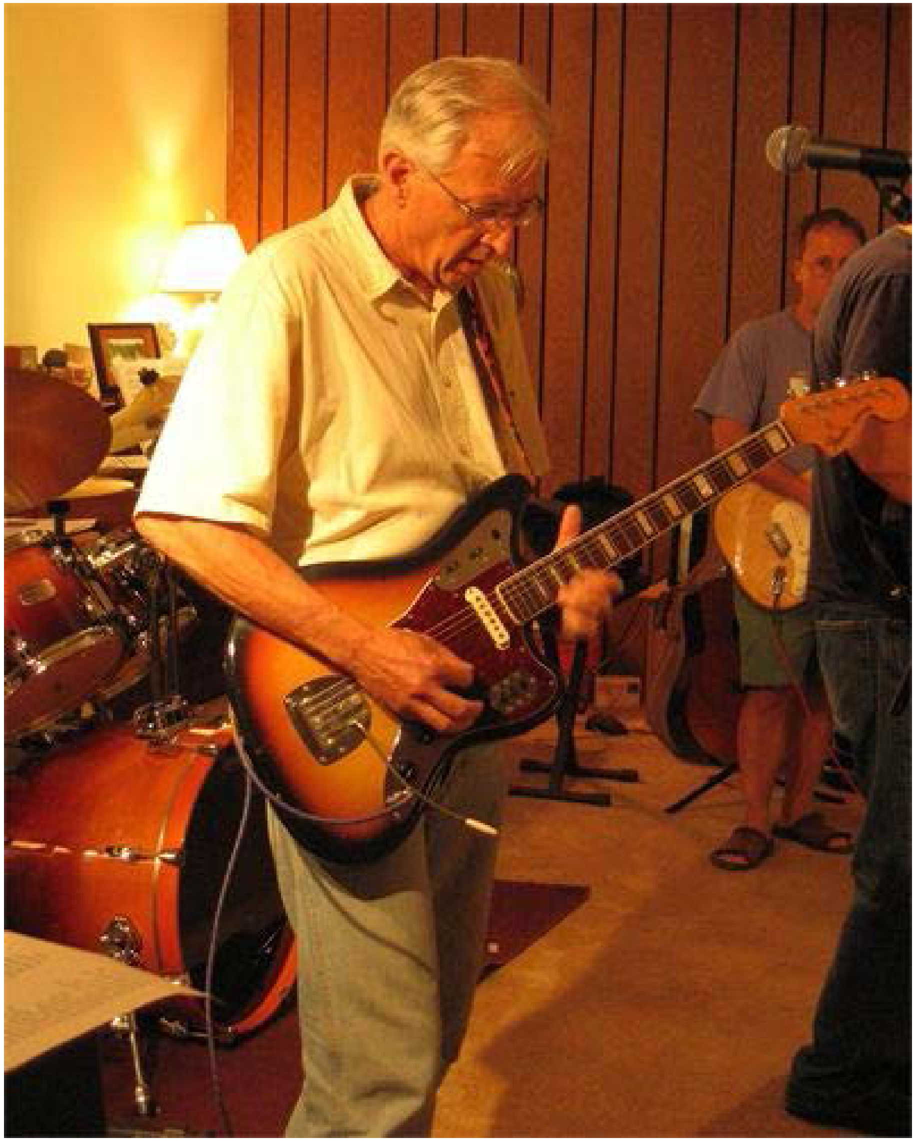
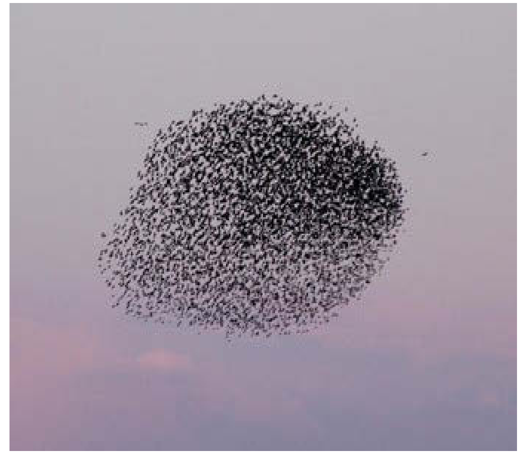
- Prof. Rajarshi Roy (UMD)
- Aaron Hagerstrom *(Graduate Research Assistant, Physics)*



Outline

- Synchronization of Dynamical Systems
- Describing Networks
  - Master Stability Function
- Spatio-Temporal Optical Network
- Symmetries and Clusters
- Isolated Desynchronization

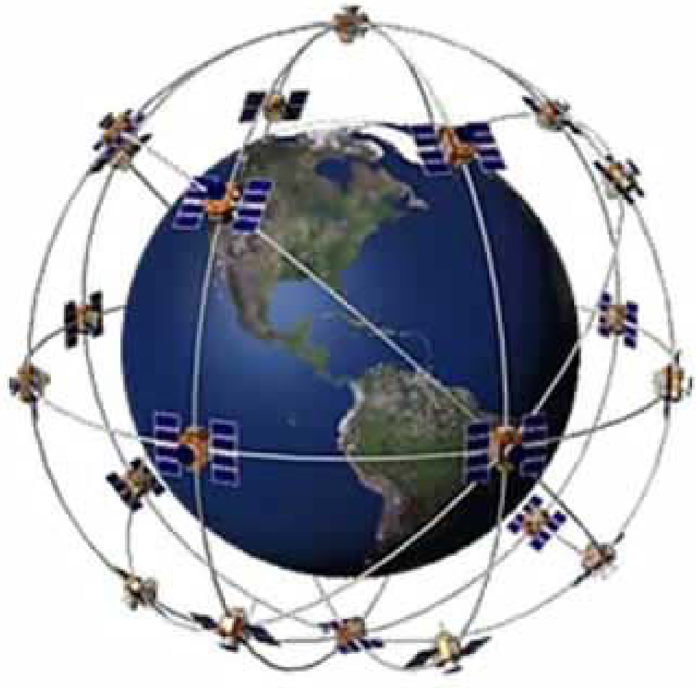
Synchronization in Nature



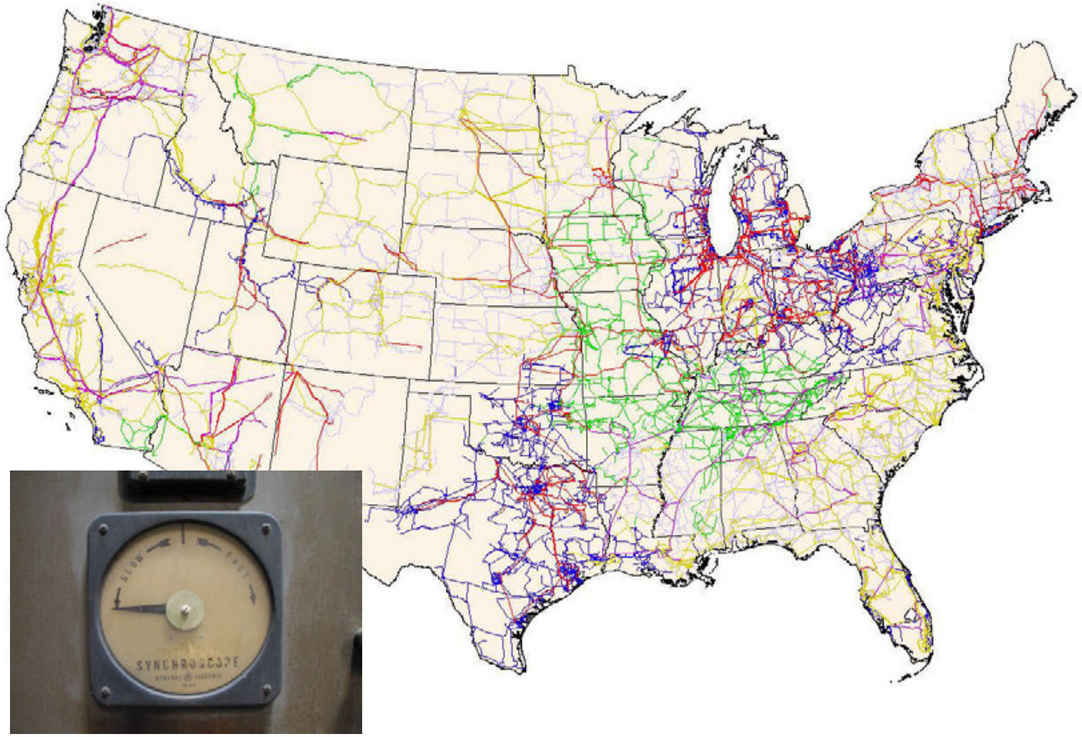
S. H. Strogatz *et al*. Nature **438**, 43 (2005).

Synchronization in Engineered Systems

GPS



Power Grid



http://en.wikipedia.org/wiki/Synchroscope

Chaotic Systems
Sensitivity to Initial Conditions

$$\frac{dx_1}{dt} = \sigma(y_1 - x_1)$$

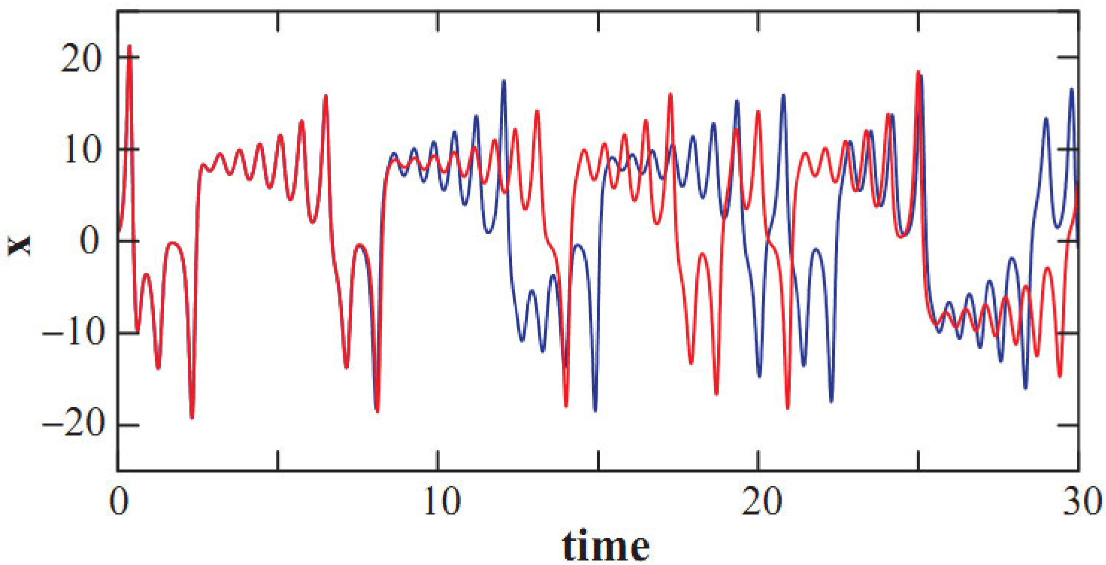
$$\frac{dy_1}{dt} = x_1(\rho - z_1) - y_1$$

$$\frac{dz_1}{dt} = x_1 y_1 - \beta z_1$$

$$\sigma = 10,\ \rho = 28,\ \beta = 8/3.$$

$$x_1(0) = 1.0$$
$$y_1(0) = 1.0$$
$$z_1(0) = 1.0$$

$$x_1(0) = 1.001$$
$$y_1(0) = 1.0$$
$$z_1(0) = 1.0$$



E. N. Lorenz, J. Atmos. Sci. **20**,130 (1963).

Synchronization of Chaos

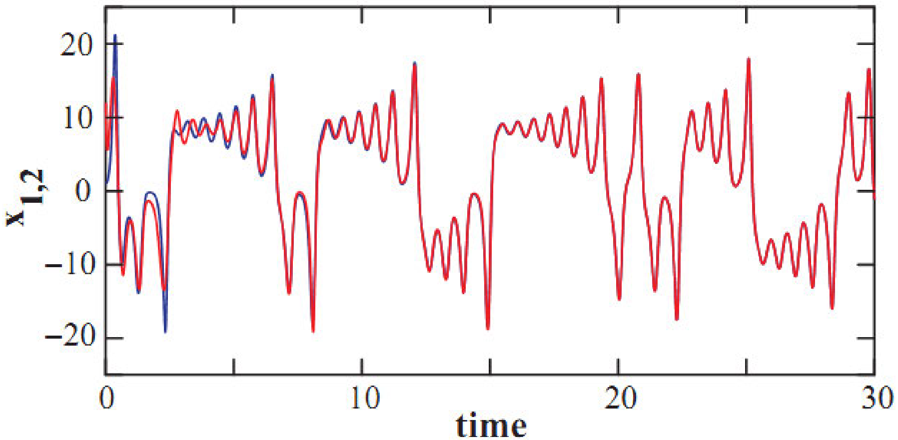
$$\frac{dx_2}{dt} = \sigma(y_2 - x_2) + \boxed{1.5(x_1 - x_2)}$$

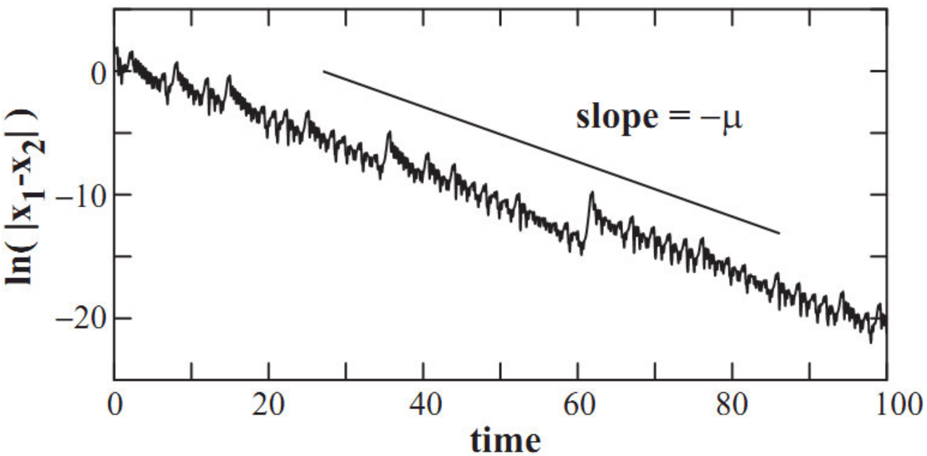
$$\frac{dy_2}{dt} = x_2(\rho - z_2) - y_2$$

$$\frac{dz_2}{dt} = x_2 y_2 - \beta z_2$$

$$x_1(0) = 1.0$$
$$y_1(0) = 1.0$$
$$z_1(0) = 1.0$$

$$x_2(0) = 12.0$$
$$y_2(0) = 1.0$$
$$z_2(0) = 5.0$$





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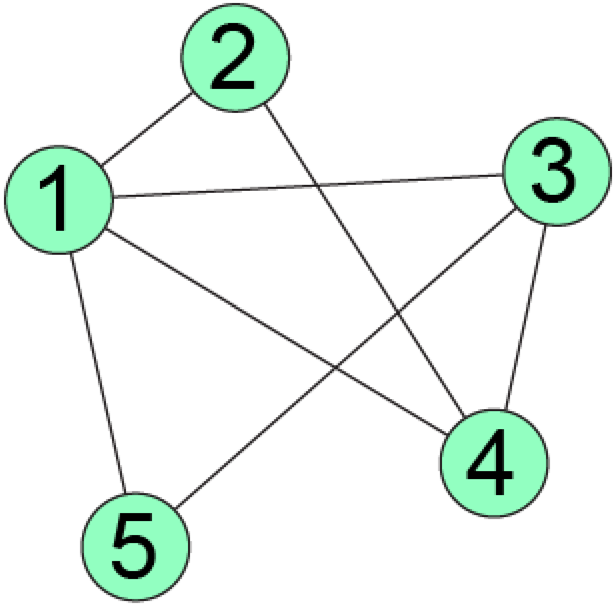
Representing Networks and Graphs



$$C = \begin{bmatrix} \bullet & 1 & 1 & 1 & 1 \\ 1 & \bullet & 0 & 1 & 0 \\ 1 & 0 & \bullet & 1 & 1 \\ 1 & 1 & 1 & \bullet & 0 \\ 1 & 0 & 1 & 0 & \bullet \end{bmatrix}$$

- $C_{ij} = 1$, if node i and j are connected
- Assume all connections are identical, bidirectional
- Generalizations:
  - Weighted connections
  - Directional links ($C_{ij} \neq C_{ji}$)

Coupled Dynamical Systems

Continuous-time:

$$\frac{d}{dt}x_i(t) = F\left(x_i(t)\right) + \sum_{j=1}^{N} C_{ij} H\left(x_j(t)\right)$$

Discrete-time:

$$x_i[n+1] = F\left(x_i[n]\right) + \sum_{j=1}^{N} C_{ij} H(x_j[n])$$

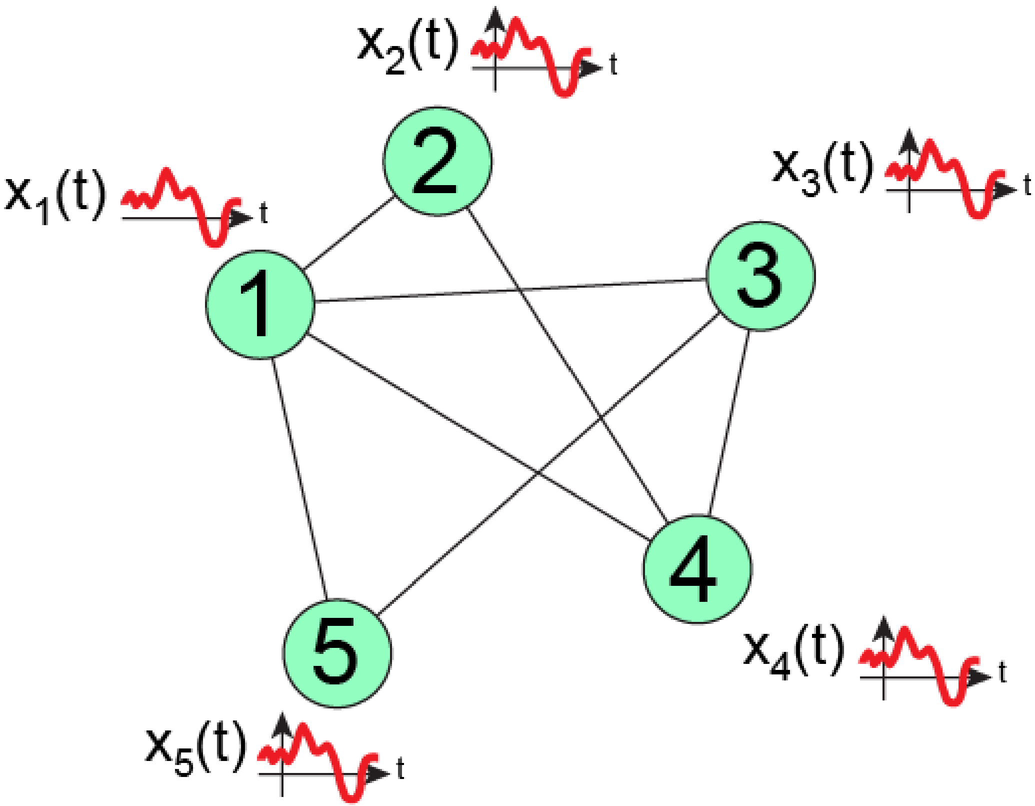
Q1: ***Can*** these equations synchronize?

(Do they *admit* a synchronous solution $x_1 = x_2 = \ldots x_N$?)

Q2: ***Do*** these equations synchronize?

(... and is the synchronous solution stable?)

Synchronization of Coupled Systems



Laplacian Coupling Matrix (row sum = 0):

$$C = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 1 & 0 \\ 1 & 0 & -3 & 1 & 1 \\ 1 & 1 & 1 & -3 & 0 \\ 1 & 0 & 1 & 0 & -2 \end{bmatrix}$$

$$x_1(t) = x_2(t) = \ldots \equiv x_s(t), \qquad \frac{d}{dt}x_s(t) = F\left(x_s(t)\right)$$

Master Stability Function

Is the Synchronous Solution Stable

VOLUME 80, NUMBER 10 PHYSICAL REVIEW LETTERS 9 MARCH 1998

Master Stability Functions for Synchronized Coupled Systems

Louis M. Pecora and Thomas L. Carroll

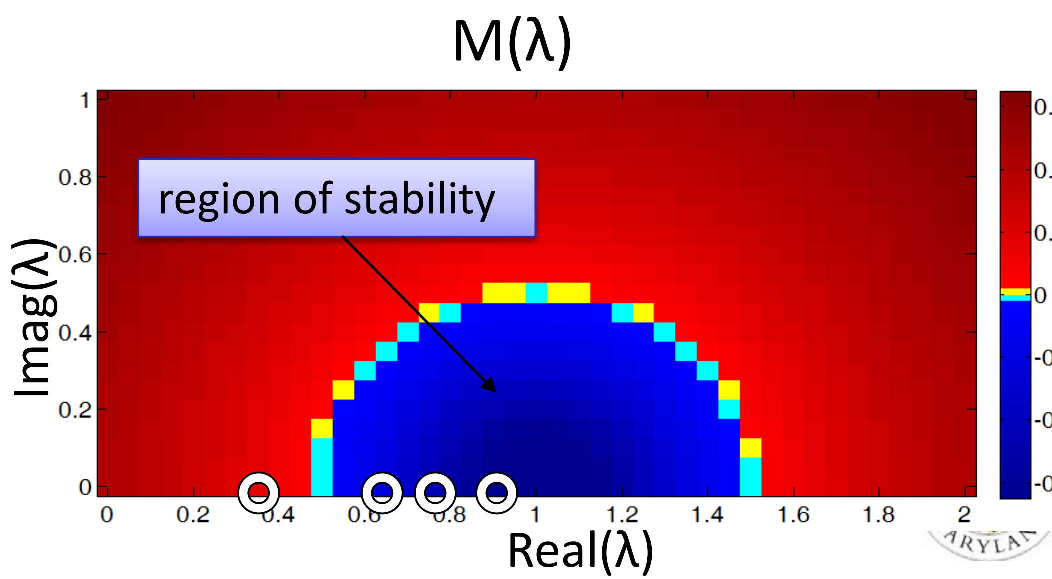
Code 6343, Naval Research Laboratory, Washington, D.C. 20375

(Received 7 July 1997)

We show that many coupled oscillator array configurations considered in the literature can be put into a simple form so that determining the stability of the synchronous state can be done by a master stability function, which can be tailored to one's choice of stability requirement. This solves, once and for all, the problem of synchronous stability for any linear coupling of that oscillator.

- Eigenvalues of C: $\{0, \lambda_1, \lambda_2, \lambda_3, \ldots\}$
- Stability condition: $M(\lambda_i) < 0$, for all i

Master Stability Function



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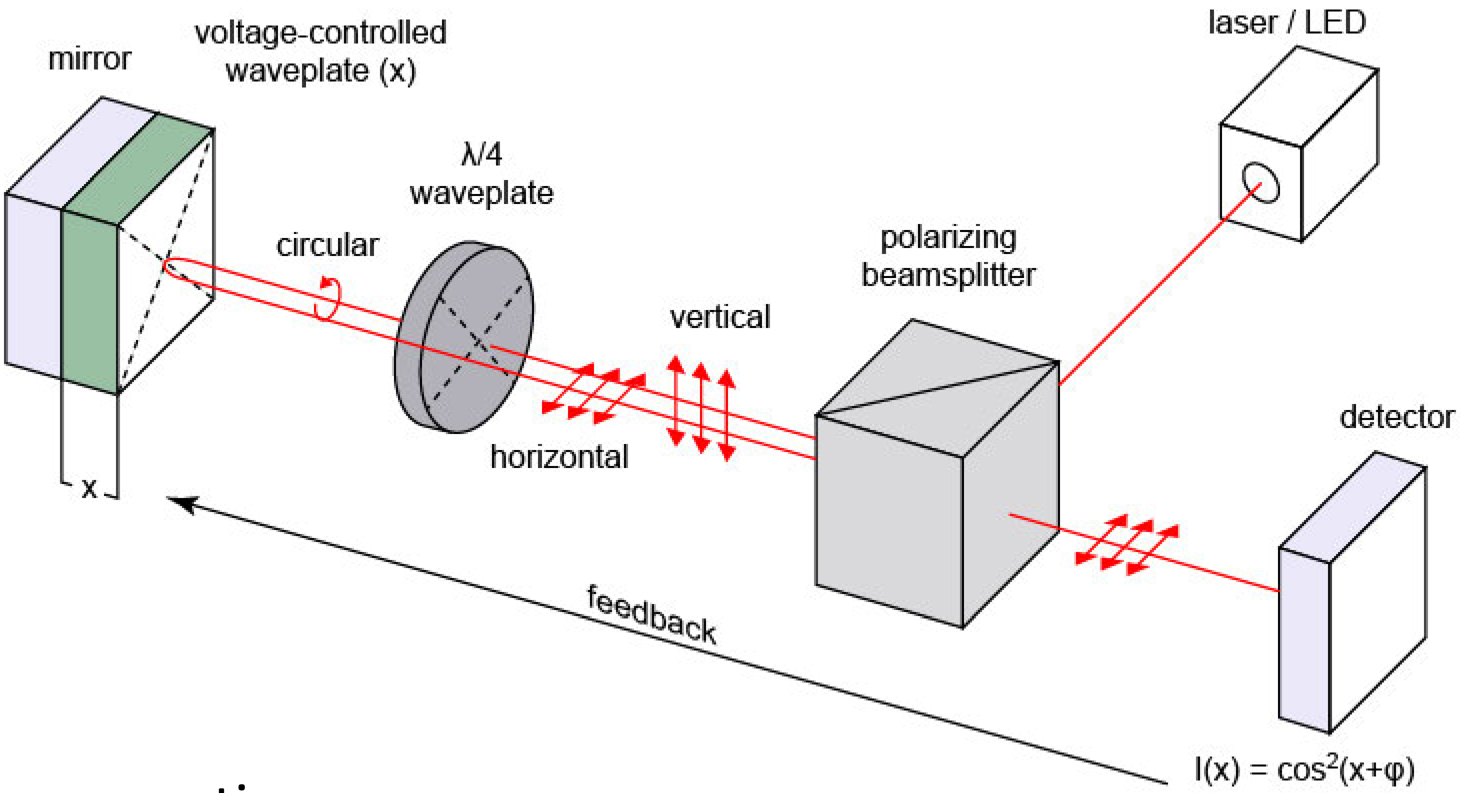
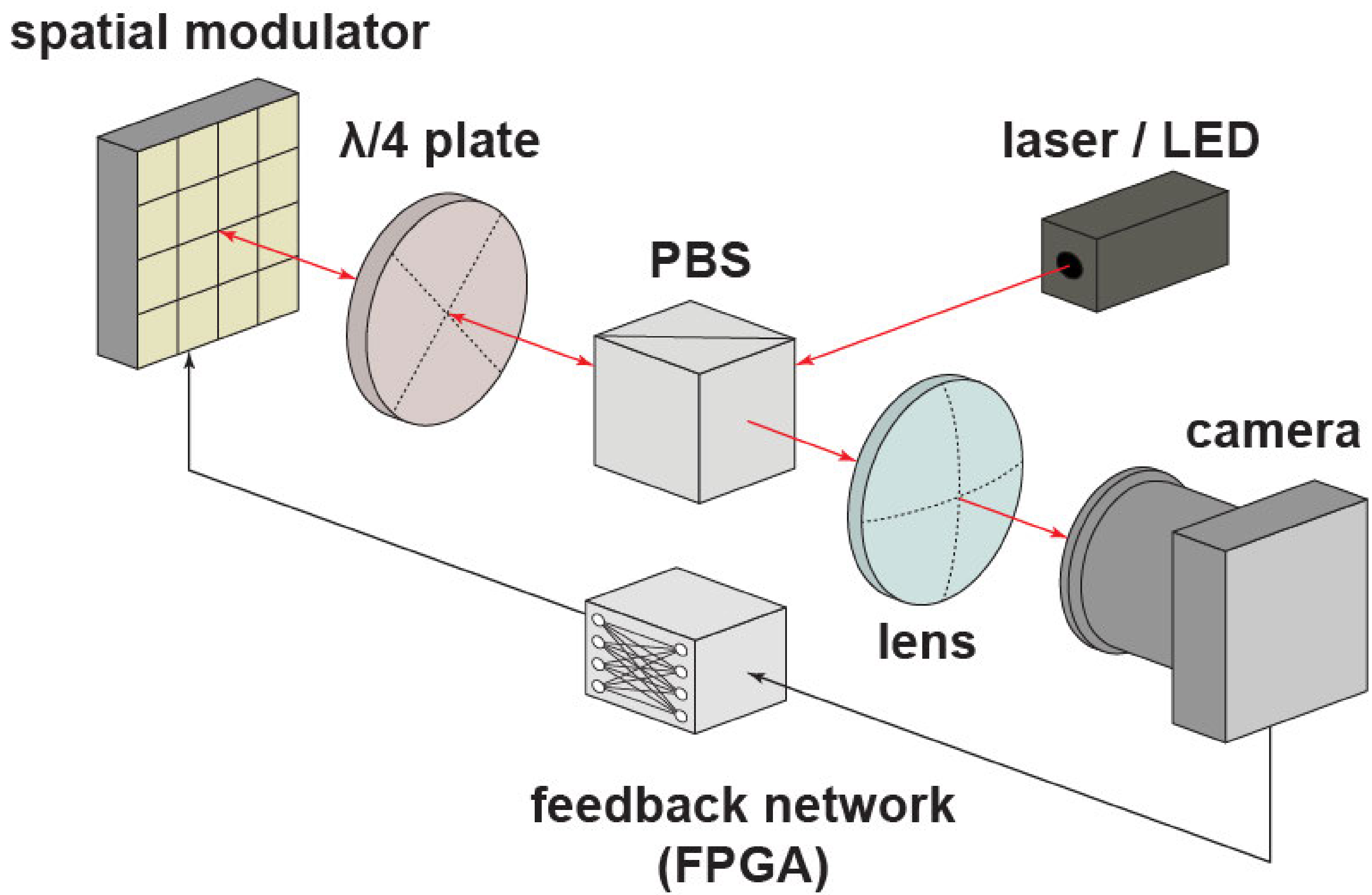
Electrooptic Feedback Loop



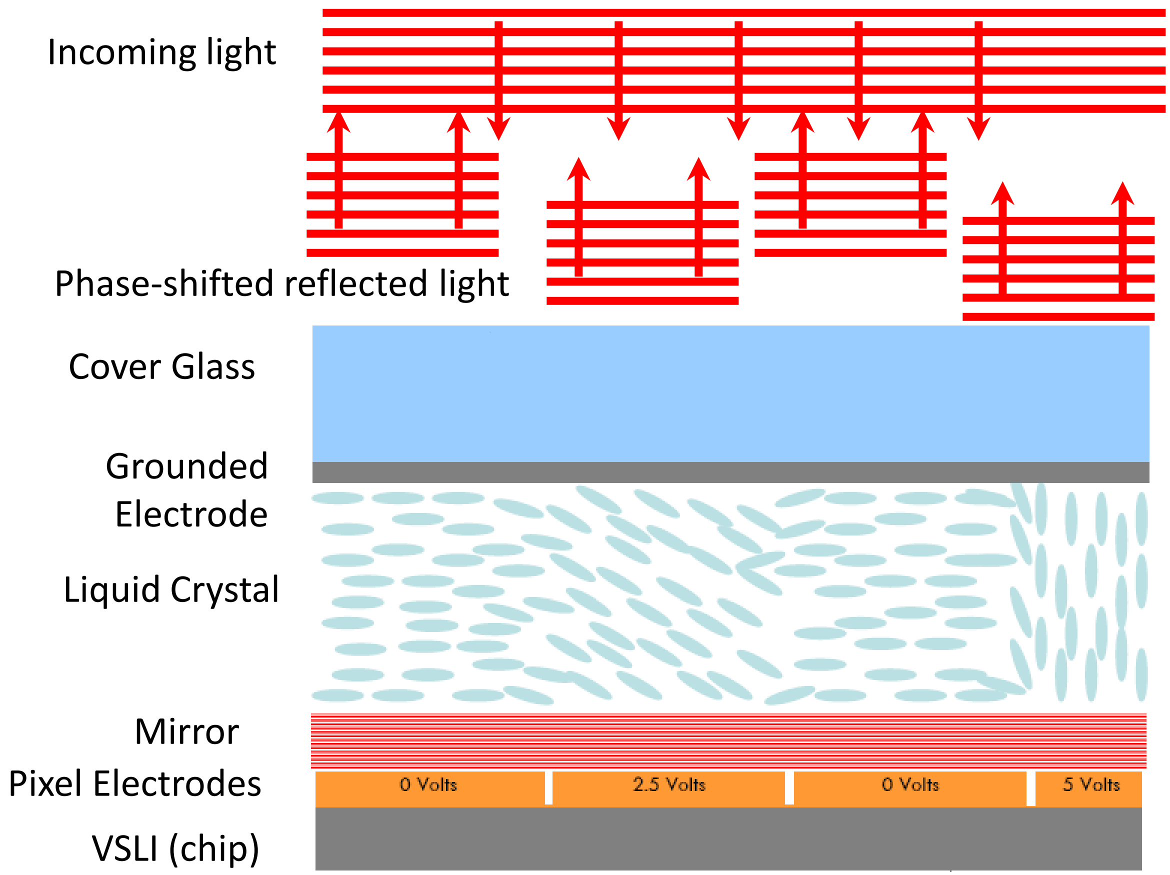
Map equation:

$$x[n+1] = a\sin^2(x) + \delta$$

Spatio-Temporal Optical Network
Video Feedback Network

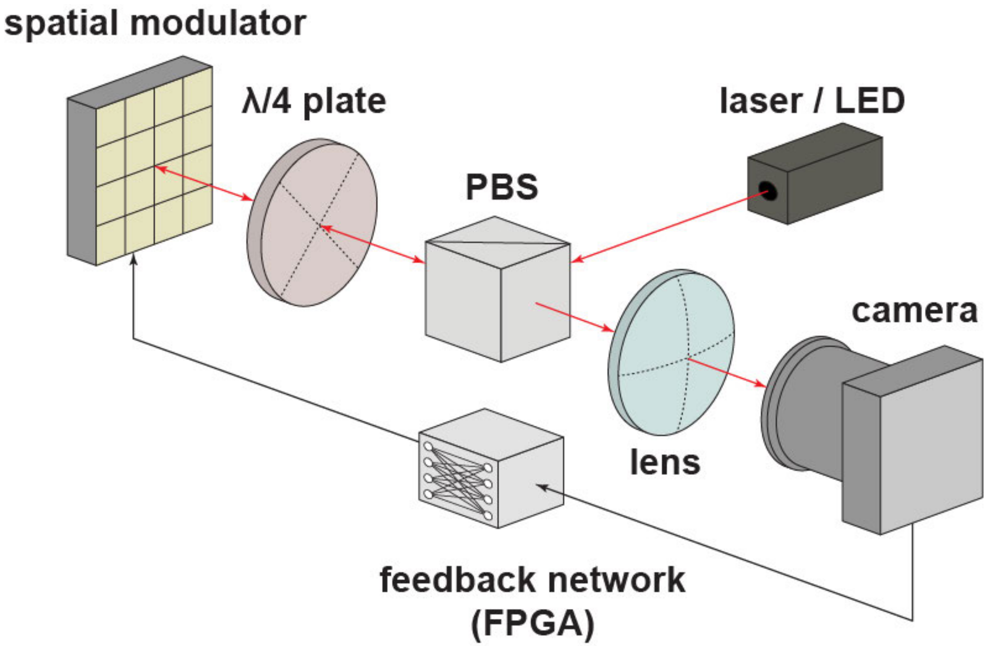


Spatial Light Phase Modulator



- Same technology used in LCD displays

Coupled Dynamical Systems



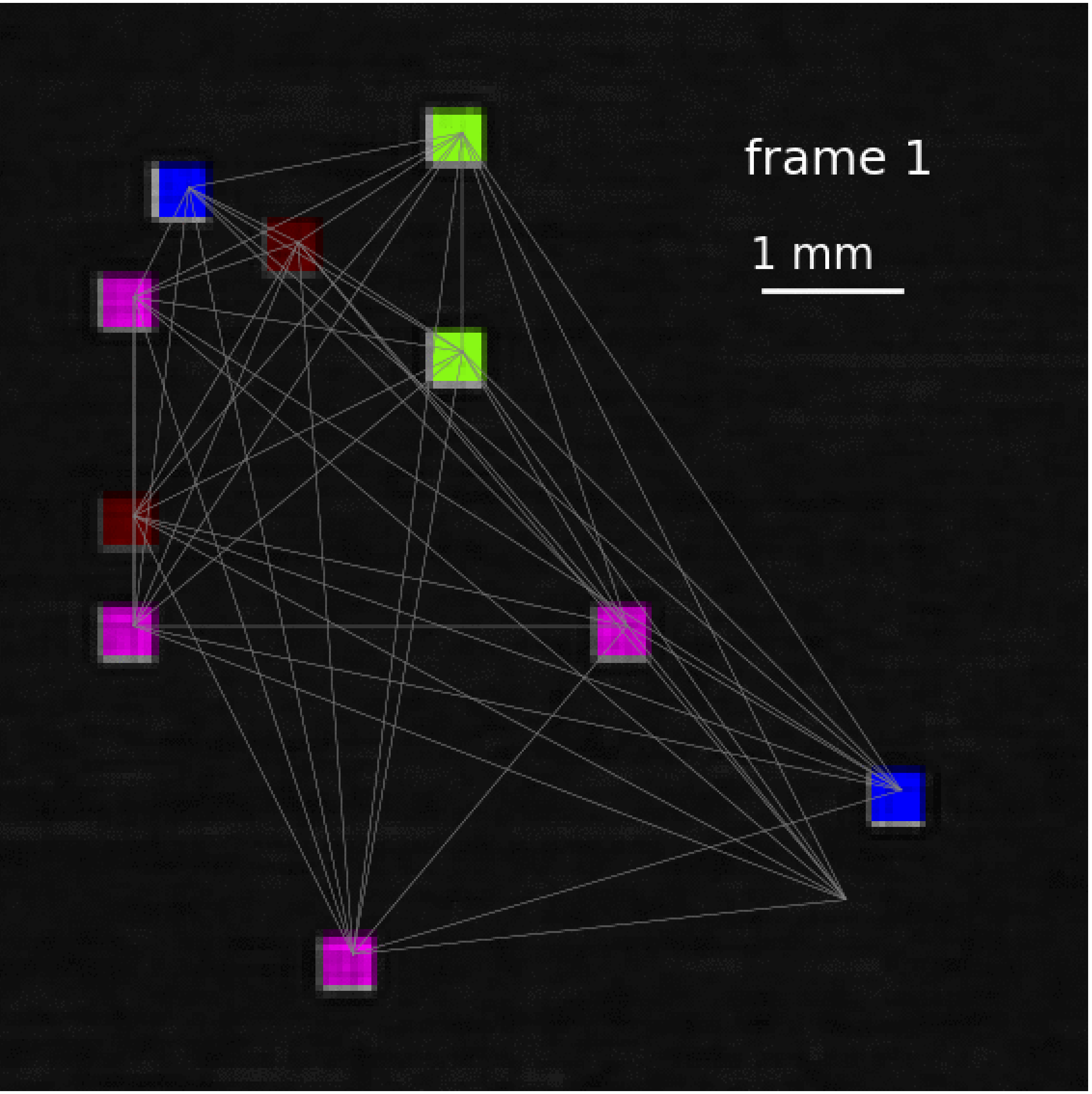
- C_{ij} programmed through feedback (or by Fourier optics)
- SLM pixels are imaged onto camera pixels
- Almost arbitrary networks can be formed

- Coupled Map Equations:

I(x) = a sin²(x)

$$x_i[n+1] = I\left(x_i[n]\right) + \sigma \sum_{j=1}^{N} C_{ij} I\left(x_j[n]\right) + \delta$$

Example: 11 node network
(6 links removed)



Network connections indicated by lines

Square patches of pixels for each node

Q: Can we predict and explain this cluster synchronization?

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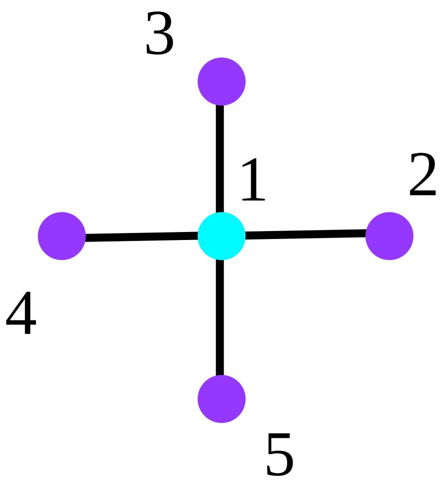
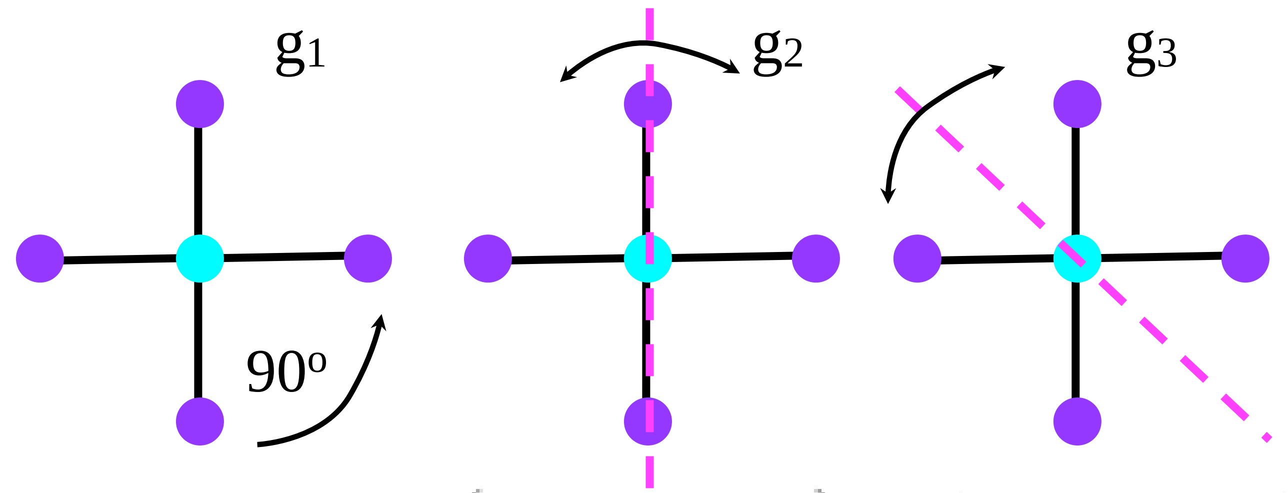
Identifying Clusters and Symmetries





$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{R}_{g_1} \qquad \boldsymbol{R}_{g_2} \qquad \boldsymbol{R}_{g_3}$$

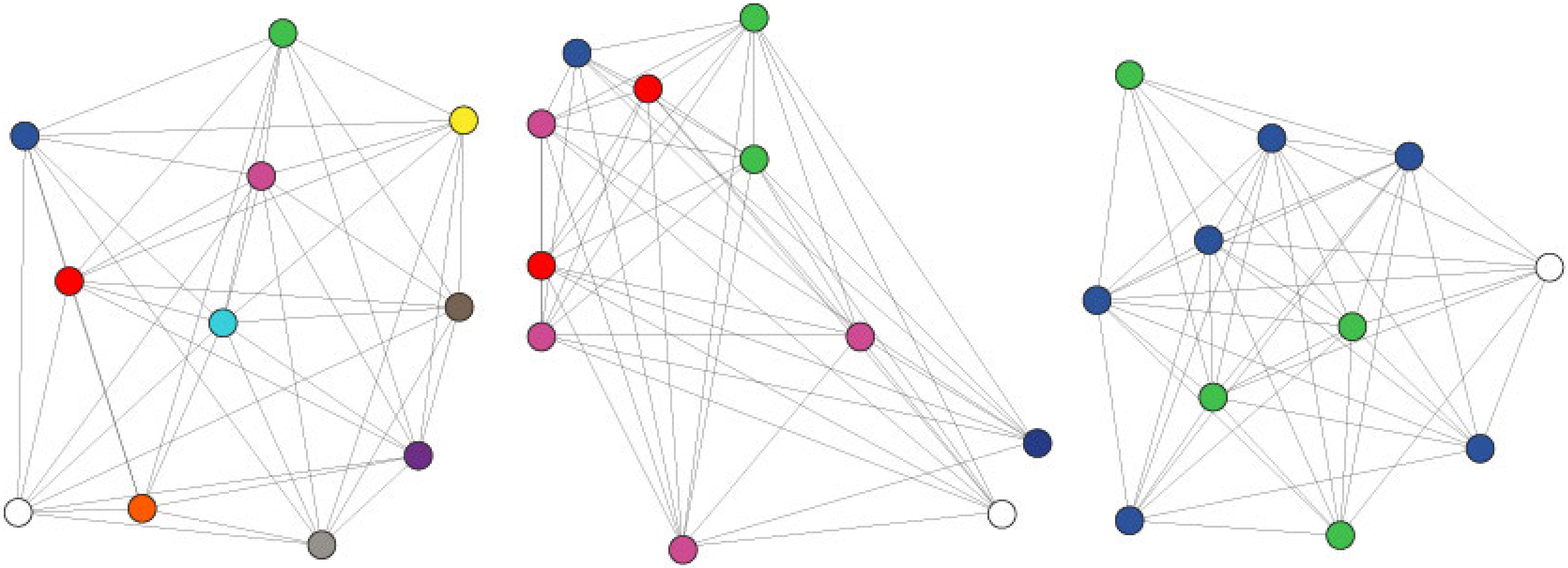
group $\mathcal{G} = \{g_i\}$

$\{\boldsymbol{R}_{g_i}\}$ Representation of the group $\mathcal{G}$

Symmetries and Dynamics

- Each symmetry can be described by a N-dimensional permutation matrix R_g
- The permutation matrix commutes with C: $R_g C = C R_g$
- The equations of motion are invariant under symmetry operation
- Orbits = subsets of nodes that permute among themselves under symmetry group (clusters!)

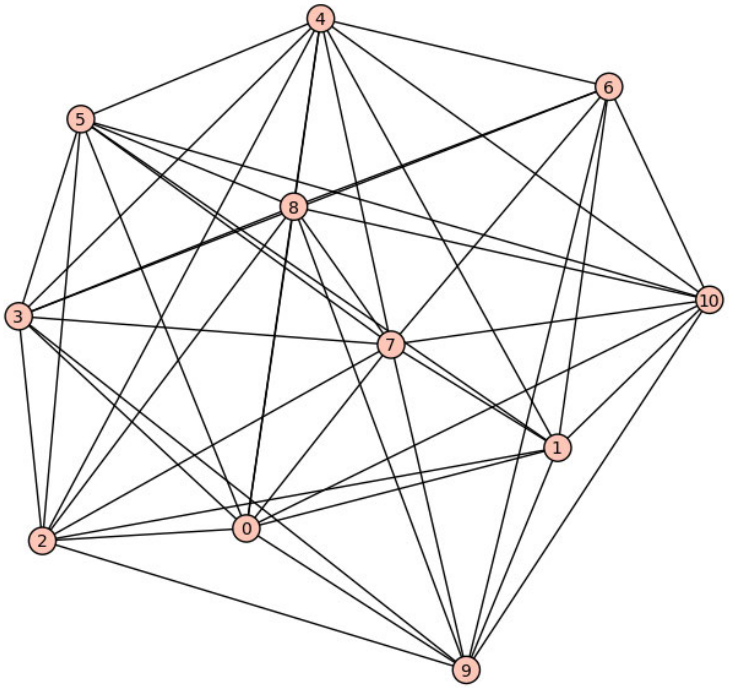
Symmetries (Example)



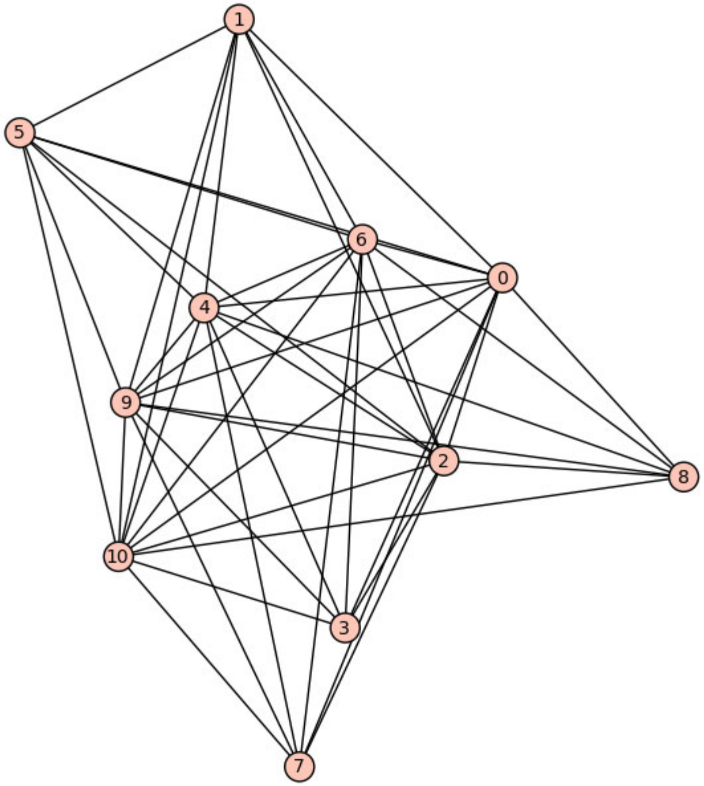
0 symmetries (11 clusters) | 32 symmetries (5 clusters) | 5,760 symmetries (3 clusters)

- Symmetries and clusters are hard to identify in all but the simplest networks!

Hidden Symmetries



0 symmetries



8640 symmetries

G.gens()= [(7,10), (6,7), (5,6), (4,8), (2,4)(8,9),(1,5), (1,11)]

(Free) Tools for Computing Symmetries

- **GAP** = Groups, Algorithms, Programming (software for computational discrete algebra) http://www.gap-system.org/
- **Sage** = Unified interface to 100's of open-source mathematical software packages, including GAP http://www.sagemath.org/
- **Python** = Open-source, multi-platform programming language http://www.python.org/

Example Output (GAP/Sage)

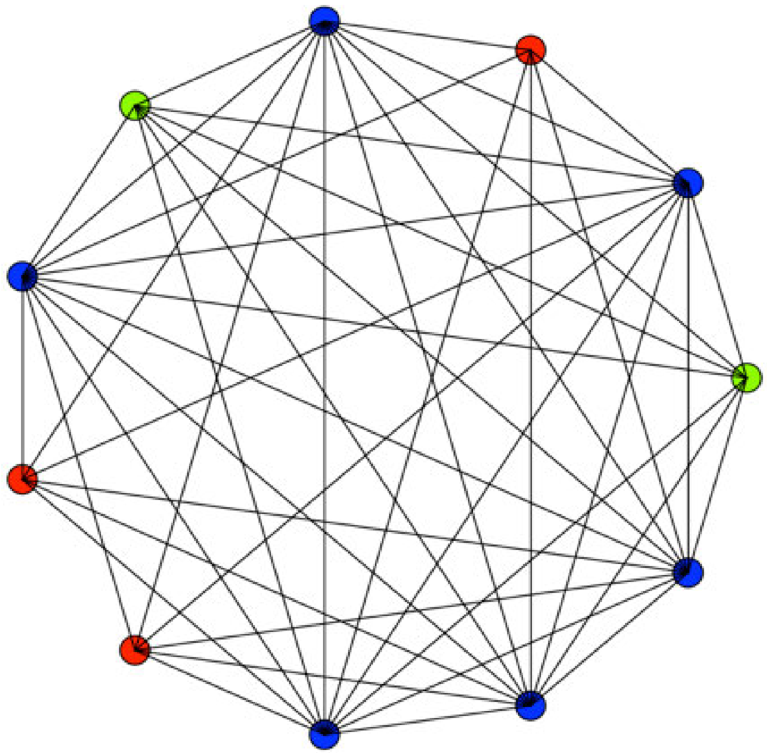
```
G.order(), G.gens()= 8640 [(9,10), (7,8), (6,9), (4,6), (3,7), (2,4), (2,11), (1,5)]

node sync vectors:
  Node 2

orb= [1, 5]
nodeSyncvec  [0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0]
cycleSyncvec  [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]
  Node 1

orb= [2, 4, 11, 6, 9, 10]
nodeSyncvec  [1, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1]
cycleSyncvec  [0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1]
  Node 4

orb= [3, 7, 8]
nodeSyncvec  [0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0]
cycleSyncvec  [0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0]
```

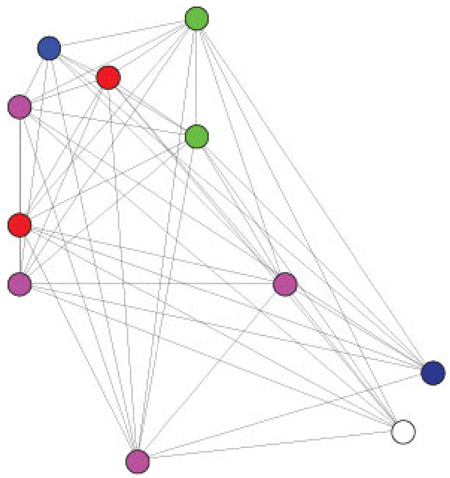


Stability of Synchronization
linearizing about cluster states

- C = coupling matrix in "node" coordinate system
- T = unitary transformation matrix to convert to IRR coordinate system
- B = TCT^{-1} = block-diagononalized form

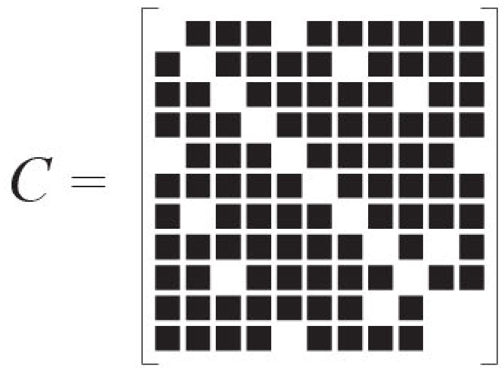
Transformed Coordinate System

for perturbations away from synchrony

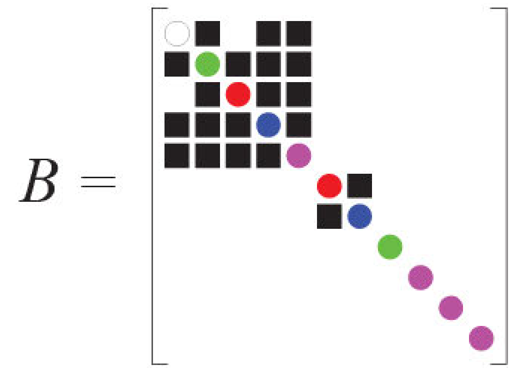


32 symmetries
(5 clusters)

"node" coordinate frame

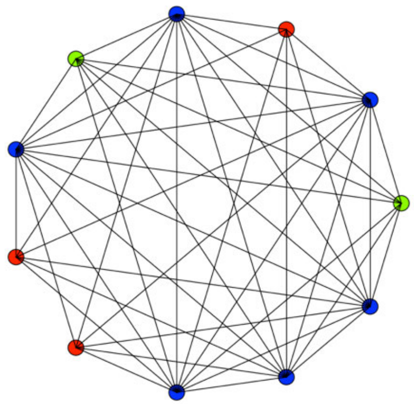


"IRR" coordinate frame



- T is not an eigendecomposition or permutation matrix
- T is found using irreducible representations (IRR) of symmetry group (computed from GAP)

Example: Diagonalization



Variational coupling matrix TCT^{-1}

-6.00	-3.46	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.(
-3.46	-5.00	-4.24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	-4.24	-6.00	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	-8.00	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	-6.00	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	-6.00	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	-11.00	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	-11.00	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-11.00	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-11.00	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-11.00

Synchronization Manifold

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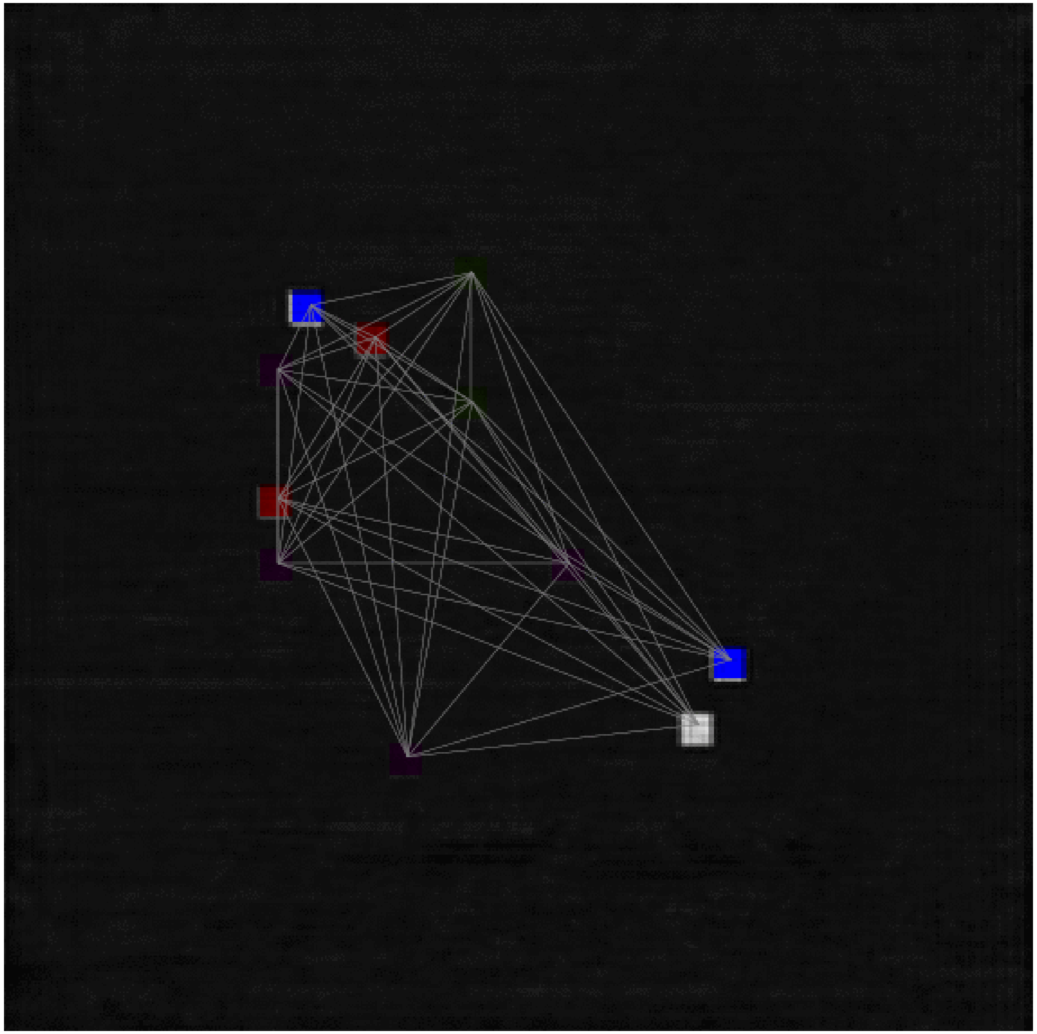
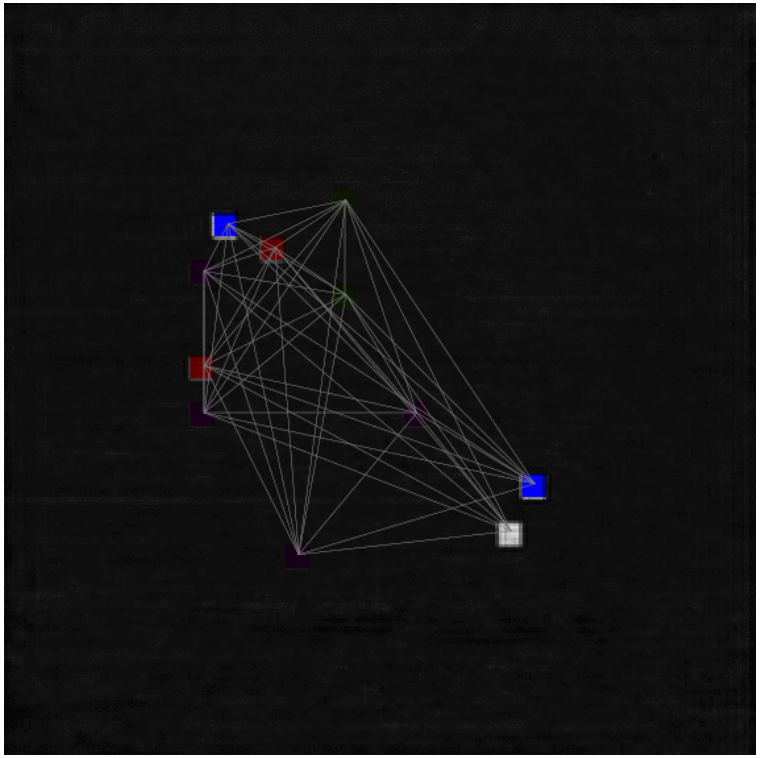
Cluster Synchronization in Experiment



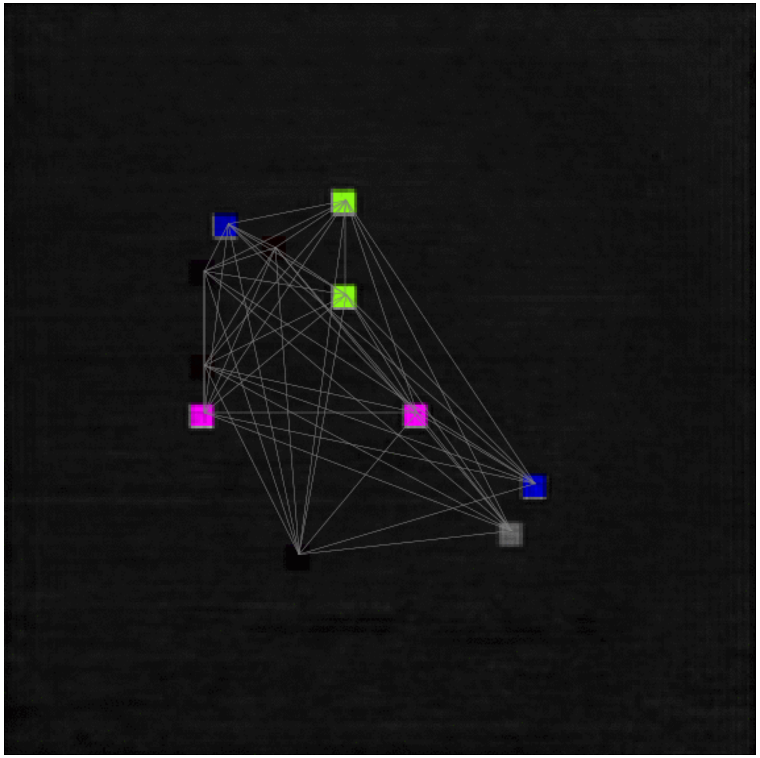
- 11 nodes
- 49 links
- 32 symmetries
- 5 clusters:
  - Blue (2)
  - Red (2)
  - Green (2)
  - Magenta (4)
  - White (1)

Isolated Desynchronization

- Pay attention to the magenta cluster:

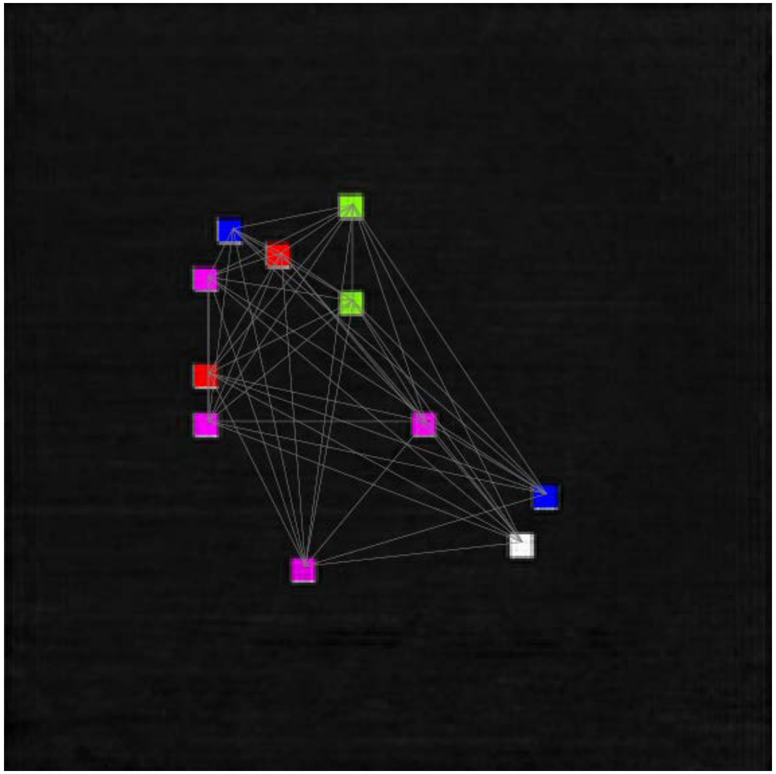


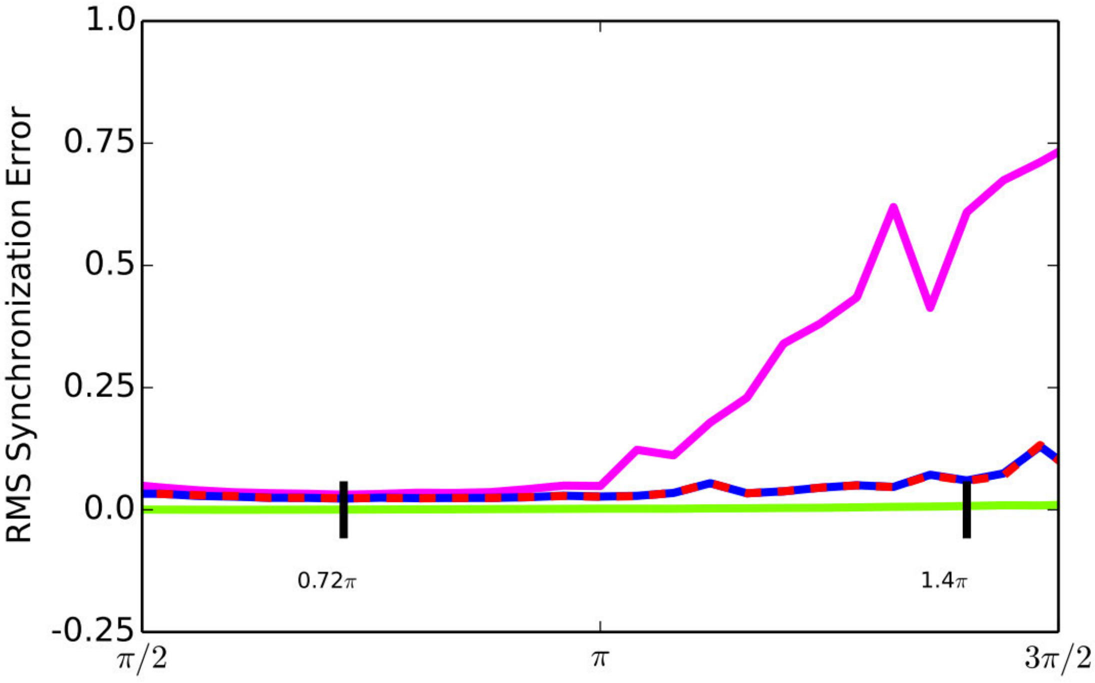
a = 0.7π



a = 1.4π

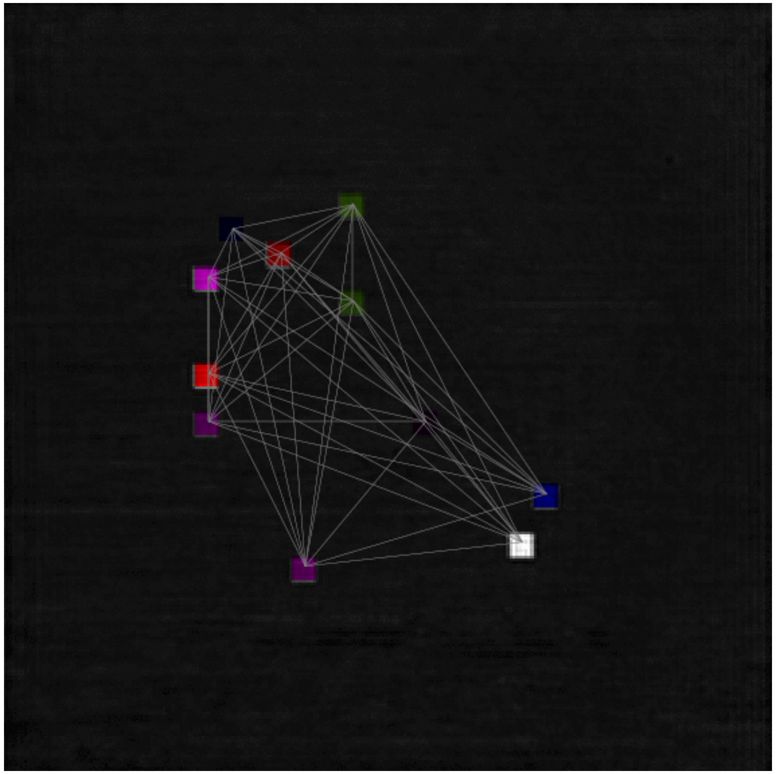
Synchronization Error

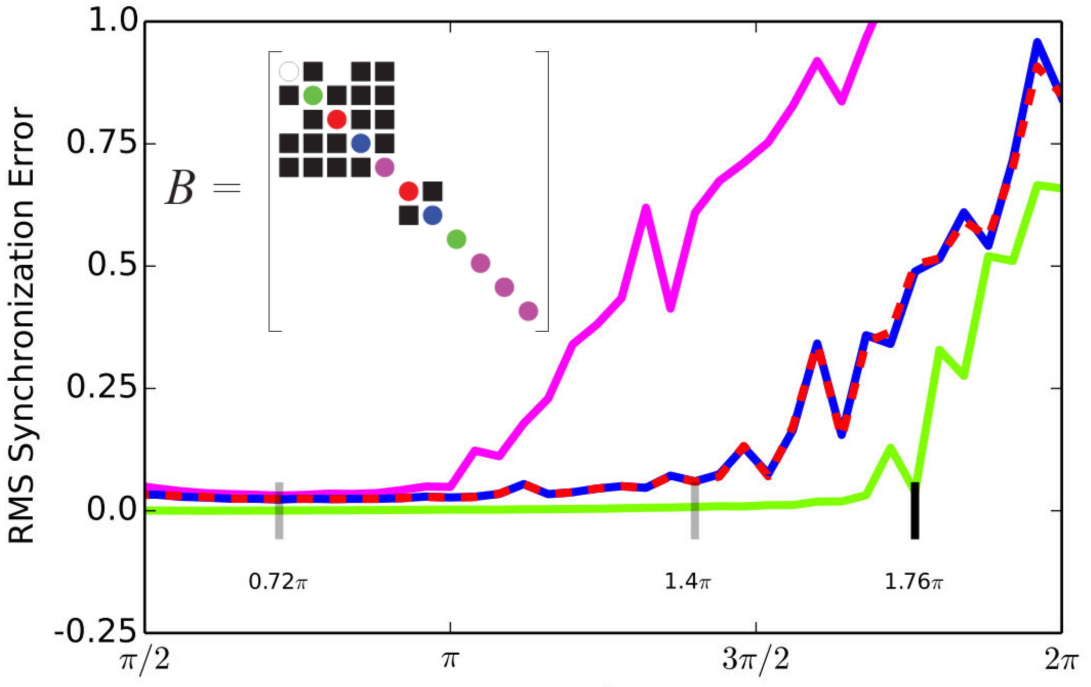




a

Intertwined Clusters

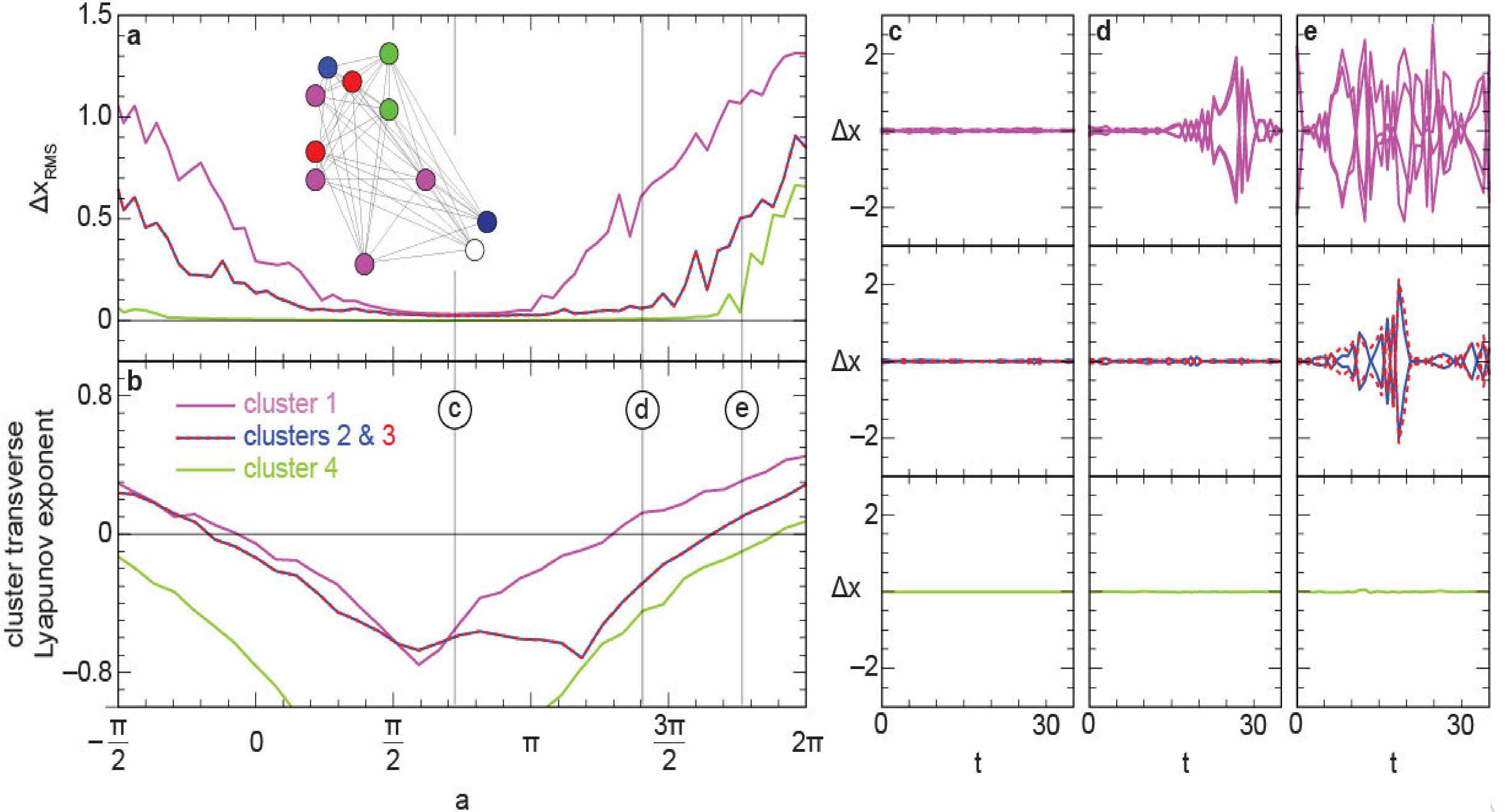




a

- Red and blue clusters are inter-dependent
- (sub-group decomposition)

Transverse Lyapunov Exponent
(linearizing about cluster synchrony)

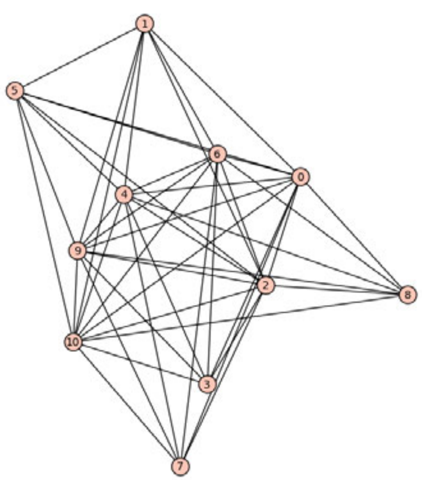


Symmetries and Clusters in Random Networks

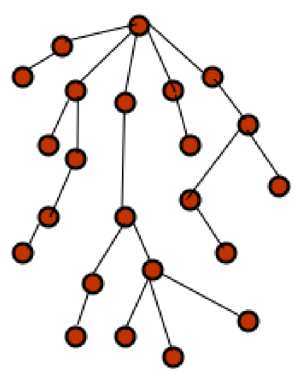
- N= 25 nodes (oscillators)
- 10,000 realizations of each type
- Calculate # of symmetries, clusters

Random

$n_{\text{delete}} = 20$

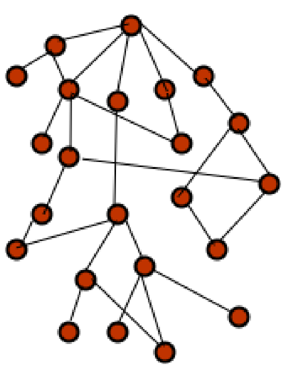


Scale-free Tree



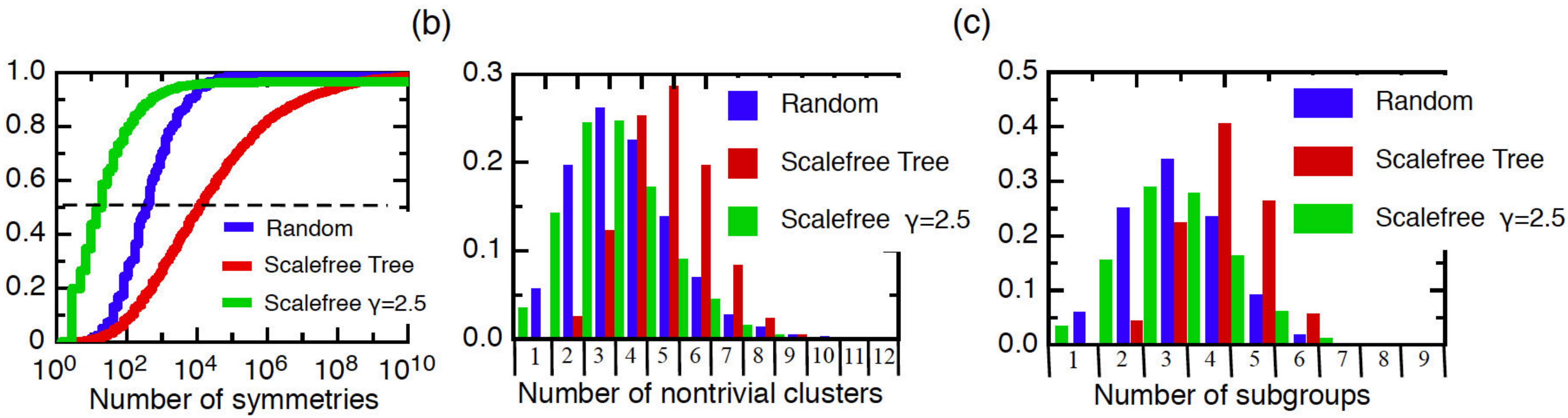
A.-L. Barabasi and R. Albert, "Emergence of scaling in random networks," *Science* **286**, 509-512 (1999).

Scale-free γ



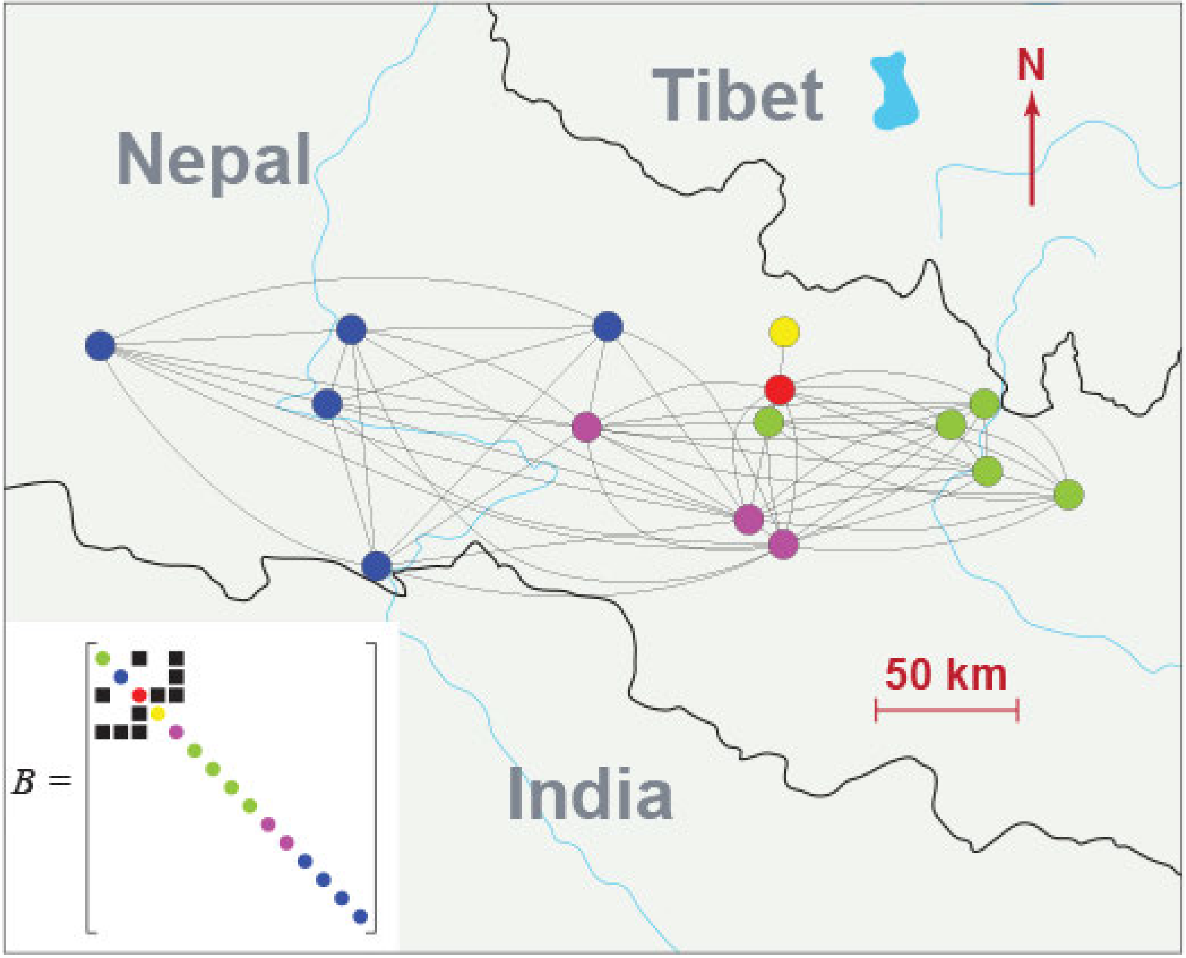
K-I Goh, B Kahng, and D Kim, "Universal behavior of load distribution in scale-free networks," *Phys. Rev. Lett.* **87**, 278701 (2001).

Symmetry Statistics

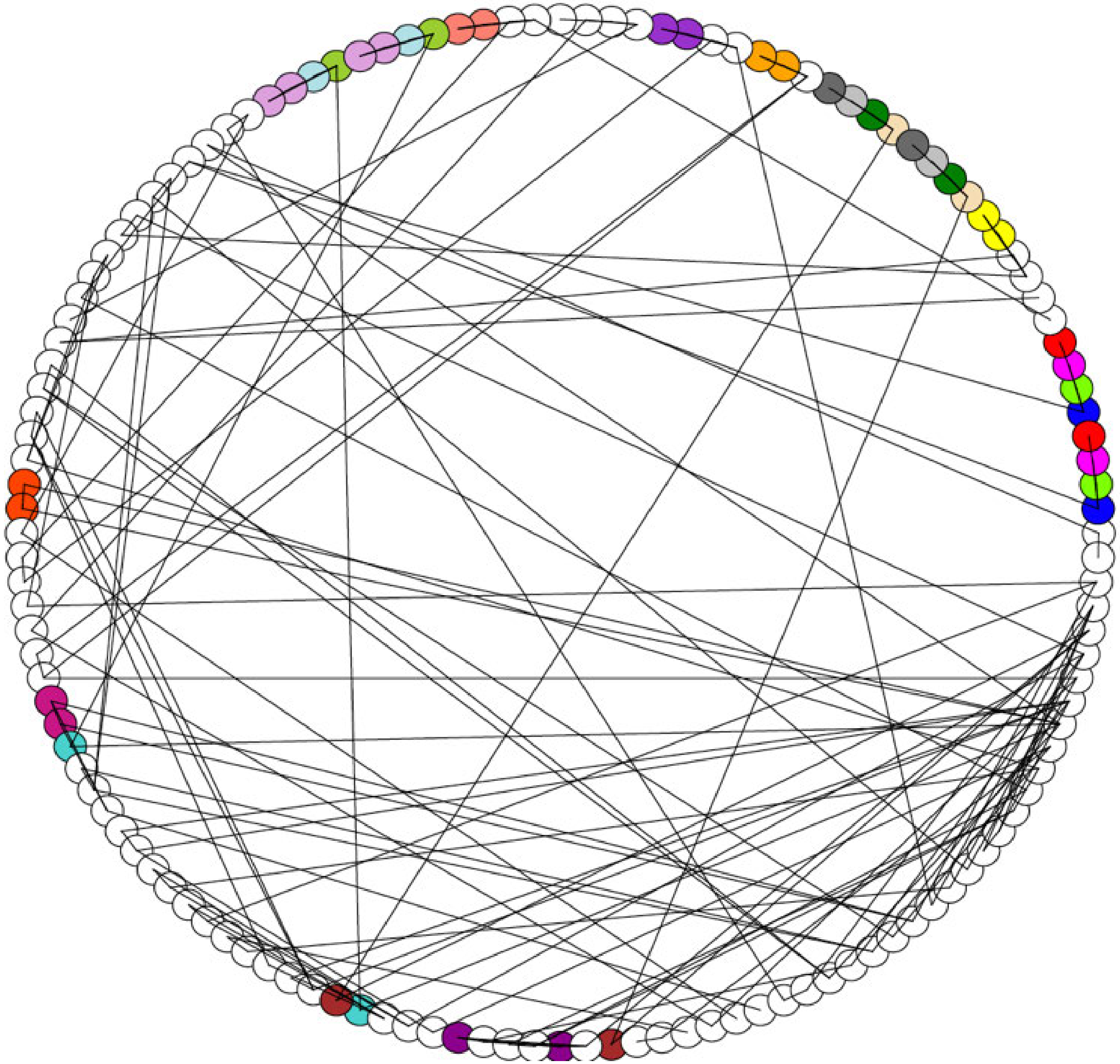


Symmetries, clusters and subgroup decompositions seem to be universal across many network models

Power Network of Nepal



Mesa Del Sol Electrical Network



- 4096 symmetries
- 132 Nodes
- 20 clusters
- 90 trivial clusters
- 10 subgroups

Symmetries & Clusters in Larger Networks

MacArthur *et al.*, “On automorphism groups of networks," *Discrete Appl. Math.* **156**, 3525 (2008).

Network	Number of Nodes $N_{\mathcal{G}}$	Number of Edges $M_{\mathcal{G}}$	Number of Symmetries $a_{\mathcal{G}}$
Human B Cell Genetic Interactions[3]	5,930	64,645	5.9374×10^{13}
C. elegans Genetic Interactions[26]	2,060	18,000	6.9985×10^{161}
BioGRID datasets[23]:			
Human	7,013	20,587	1.2607×10^{485}
S. cerevisiae	5,295	50,723	6.8622×10^{64}
Drosophila	7,371	25,043	3.0687×10^{493}
Mus musculus	209	393	5.3481×10^{125}
Internet (Autonomous Systems Level)[12]	22,332	45,392	$1.2822 \times 10^{11,298}$
US Power Grid[25]	4,941	6,594	5.1851×10^{152}

> 88% of nodes are in clusters in all above networks

Summary

- Synchronization is a widespread in both natural and engineered systems
- Many systems exhibit patterns or clusters of synchrony
- Synchronization patterns are intimately connected to the hidden symmetries of the network

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